

MINIMAX POWER OF RAO'S SCORE TEST FOR INDEPENDENCE IN HIGH DIMENSIONS

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ABSTRACT. Let \mathbf{R} be the Pearson correlation matrix of m normal random variables. The Rao's score test for the independence hypothesis $H_0 : \mathbf{R} = \mathbf{I}_m$, where \mathbf{I}_m is the identity matrix of dimension m , was first considered by Schott (2005) in the high dimensional setting. In this paper, we study the asymptotic minimax power function of this test, under an asymptotic regime in which both m and the sample size n tend to infinity with the ratio m/n upper bounded by a constant. In particular, our result implies that the Rao's score test is rate-optimal for detecting the dependency signal $\|\mathbf{R} - \mathbf{I}_m\|_F$ of order $\sqrt{m/n}$, where $\|\cdot\|_F$ is the matrix Frobenius norm.

1. INTRODUCTION

Let $(X_1, \dots, X_m)'$ be an m -variate normal vector with population Pearson correlation matrix denoted by $\mathbf{R} = (\rho_{pq})_{1 \leq p, q \leq m}$. Suppose we observe n independent samples X_{p1}, \dots, X_{pn} for each component X_p , $1 \leq p \leq m$. When the dimension m can be larger than the sample size n , Schott (2005) was the first to consider the Rao's score statistic

$$(1.1) \quad T = \sum_{1 \leq p < q \leq m} \hat{\rho}_{pq}^2,$$

for testing the independence null hypothesis

$$(1.2) \quad H_0 : \mathbf{R} = \mathbf{I}_m,$$

where $\hat{\rho}_{pq}$, $1 \leq p \neq q \leq m$ is the sample correlation of the pair (X_p, X_q) computed from the data, and \mathbf{I}_m is the m -by- m identity matrix. It was showed to be asymptotically normal under H_0 as both m and n go to infinity with the ratio m/n converging to a positive constant. The purpose of this paper is to complement the theoretical study of T by investigating its minimax power under alternatives of the form

$$H_1 : \mathbf{R} \in \Theta(b),$$

where for any constant $b > 0$ and matrix Frobenius norm $\|\cdot\|_F$, we define the set of Pearson correlation matrices

$$(1.3) \quad \Theta(b) := \{\mathbf{R} : \|\mathbf{R} - \mathbf{I}_m\|_F \geq b\sqrt{m/n}, \text{ diag}(\mathbf{R}) = \mathbf{I}_m\},$$

which comprises a composite alternative hypothesis delineated by a signal size $\|\mathbf{R} - \mathbf{I}_m\|_F$ of order no less than $\sqrt{m/n}$.

There are three major approaches to testing independence with growing dimension m in the literature, to the best of our knowledge. The first is the statistic T considered in this paper. Being a “sum” of squared pairwise sample correlation as in (1.1), it is good at detecting diffuse dependency among many pairs of variables. Such dependency is most naturally described by the signal $\|\mathbf{R} - \mathbf{I}_m\|_F$. In fact, the main result in this paper will show that T is minimax rate optimal for detecting such signal. The second approach considers the “max” statistic,

$$\max_{1 \leq p < q \leq m} \hat{\rho}_{pq}^2.$$

Following many previous works (Jiang, 2004, Li et al., 2010, 2012, Liu et al., 2008, Zhou, 2007), Cai and Jiang (2011) showed that it admits an asymptotic Gumbel distribution under H_0 in the ultra high dimensional regime when m can be as large as e^{n^c} for some constant $0 < c < 1$, as $m, n \rightarrow \infty$. Naturally, it is good at detecting a structured alternative whose population correlation matrix \mathbf{R} has sparse non-zero off-diagonal entries with considerable magnitudes. Both the “sum” and “max” approaches base their test on forming intuitive statistics that measure the overall dependency among the m variables, with their respective non-parametric extensions; see Leung and Drton (2015) and Han and Liu (2014). The third is likelihood ratio test (LRT), which is well-known to give implementable test only if the dimension m is smaller than n . Despite this limitation, Jiang and Qi (2015) showed the LRT statistic to be asymptotically normal when $m, n \rightarrow \infty$, as long as $m + 4$ is less than n .

For a justification of (1.1) as the Rao’s score, see Appendix A in Leung and Drton (2015).

2. NOTATIONS AND MAIN RESULTS

For any positive integer k , $[k]$ is defined as the set $\{1, \dots, k\}$. \mathcal{S}_k is the symmetric group of order k . Depending on the context, its elements will sometimes be treated as permutation functions on k elements, or simply permutations of the set $[k]$. C always denotes a positive constant that is universal, i.e, its value may change from place to place but does not depend on m and n . “ $a \lesssim b$ ” means that $a \leq Cb$ for some constant $C > 0$. $\mathbb{E}[\cdot]$, $\text{Var}[\cdot]$ and $P[\cdot]$ are expectation, variance and probability operators respectively.

In this paper we shall *always* assume that, for all $1 \leq p \leq m$, $\text{Var}[X_p] = 1$ and $\mathbb{E}[X_p] = 0$. Thus, for a duple $(p, q) \in [m] \times [m]$, $\mathbb{E}[X_p X_q] = \rho_{pq}$, and its corresponding squared sample correlation is defined as

$$(2.1) \quad \hat{\rho}_{pq}^2 := \frac{S_{pq}^2}{S_{pp}S_{qq}} = f(S_{pp}, S_{qq}, S_{pq}),$$

where $f : \mathbb{R}_{>0}^2 \times \mathbb{R} \rightarrow \mathbb{R}$ is the function

$$(2.2) \quad f(u_1, u_2, u_3) := u_1^{-1} u_2^{-1} u_3^2,$$

and

$$S_{pq} := \frac{\sum_{i=1}^n X_{pi} X_{qi}}{n}.$$

We will also use

$$\bar{S}_{pq} := S_{pq} - \rho_{pq}$$

to denote the centered sample covariance. Imposing these assumptions and using the seemingly more restrictive definition of sample correlations in (2.1) does not forgo any generality of our results to follow; see Anderson (2003, Theorem 3.3.2) for the related discussion.

According to Chen and Shao (2012, Theorem 2.2) who refined the asymptotic result of Schott (2005) under H_0 , for a given $\alpha \in (0, 1)$, a test of asymptotic level α based on (1.1) is given as

$$(2.3) \quad \psi = I \left(T - \frac{m(m-1)}{2n} > \frac{m}{n} z_\alpha \right),$$

where $I(\cdot)$ is the indicator function, $z_\alpha := \bar{\Phi}^{-1}(\alpha)$, and Φ and $\bar{\Phi}(x) := 1 - \Phi(x)$ are respectively the cumulative distribution function and tail probability of a standard normal variate. Below, $\mathbb{E}_{\mathbf{R}}[\cdot]$ simply emphasizes that the expectation is taken with respect to a particular correlation matrix $\mathbf{R} \in \Theta(b)$.

Theorem 2.1 (Main result: asymptotic minimax power). *Suppose $m, n \rightarrow \infty$ such that $\frac{m}{n} \leq \kappa$ for some constant $\kappa < \infty$. For any significance level $\alpha \in (0, 1)$, the asymptotic minimax power of ψ is given as*

$$\lim_{n \rightarrow \infty} \inf_{\Theta(b)} \mathbb{E}_{\mathbf{R}}[\psi] = \bar{\Phi}(z_\alpha - 2^{-1}b^2)$$

This minimax result resembles Cai and Ma (2013, Theorem 4), in which the different problem of testing $H_0 : \mathbf{\Sigma} = \mathbf{I}_m$, where $\mathbf{\Sigma}$ is the *covariance* matrix of $(X_1, \dots, X_m)'$, is studied. Despite this, Theorem 1 and Remark 1 in their paper indicate that a matching lower bound on the detectable signal size as measured by $\|\mathbf{R} - \mathbf{I}_m\|_F$ can be established for our problem (1.2), which we restate here for our readers' convenience.

Theorem 2.2 (Matching lower bound, Cai and Ma (2013)). *Let $0 < \alpha < \beta < 1$. Suppose $m, n \rightarrow \infty$ such that $\frac{m}{n} \leq \kappa$ for some constant $\kappa < \infty$. Then there exists a constant $b = b(\kappa, \beta - \alpha) < 1$, such that*

$$\limsup_{n \rightarrow \infty} \inf_{\Theta(b)} \mathbb{E}_{\mathbf{R}}[\phi] < \beta$$

for any test ϕ with significance level α for testing H_0 .

The lower bound result says that no α -level test for H_0 can achieve a preset target power if the signal size $\|\mathbf{R} - \mathbf{I}_m\|_F$ falls below a certain threshold modulo the separation rate $\sqrt{m/n}$. Our main result in Theorem 2.1 hence suggests that our test ψ is “rate” optimal when the ratio m/n is bounded, since the asymptotic minimax power $\lim_{n \rightarrow \infty} \inf_{\Theta(b)} \mathbb{E}_{\mathbf{R}}[\psi]$ tends to one as $b \rightarrow \infty$.

Although the result in Theorem 2.1 is neat, its proof, which occupies the rest of this paper, is quite involved. As will become clear later, this is because our statistic T is constructed with Pearson correlations whose higher order moment properties involve a lot of computations to be understood; see Hotelling (1953, Section 7) for classical work on this. At some point in this paper we will use `mathematica` to

help us with certain symbolic calculations. We shall begin with a Taylor expansion of the expression for $\hat{\rho}_{pq}^2$ in terms of the function f in (2.1). We need the multi-index notations: For a vector $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_k)$ of k non-negative integers, $\boldsymbol{\lambda}! = \lambda_1! \dots \lambda_k!$ and $|\boldsymbol{\lambda}| = \lambda_1 + \dots + \lambda_k$, and if $g = g(u_1, \dots, u_k)$ is a function in k arguments, $\partial^{\boldsymbol{\lambda}} g(\tilde{u}_1, \dots, \tilde{u}_k) = \frac{\partial^{|\boldsymbol{\lambda}|} g}{\partial u_1^{\lambda_1} \dots \partial u_k^{\lambda_k}} \Big|_{u_i = \tilde{u}_i}$ is its partial derivative with respect to $\boldsymbol{\lambda}$ evaluated at the point $(\tilde{u}_1, \dots, \tilde{u}_k)$. Since $\rho_{pq}^2 = f(1, 1, \rho_{pq}) = f(\rho_{pp}, \rho_{qq}, \rho_{pq})$, by Taylor's theorem, for each pair $1 \leq p \neq q \leq m$,

$$(2.4) \quad \hat{\rho}_{pq}^2 - \rho_{pq}^2 = \sum_{\substack{\boldsymbol{\lambda} \in \mathbb{N}_{\geq 0}^3: \\ 1 \leq |\boldsymbol{\lambda}| \leq 4}} \frac{\partial^{\boldsymbol{\lambda}} f(1, 1, \rho_{pq})}{\boldsymbol{\lambda}!} \bar{S}_{pp}^{\lambda_1} \bar{S}_{qq}^{\lambda_2} \bar{S}_{pq}^{\lambda_3} + III_{pq} \quad \text{a.s.,}$$

where

$$(2.5) \quad III_{pq} := \sum_{\substack{\boldsymbol{\lambda} \in \mathbb{N}_{\geq 0}^3: \\ |\boldsymbol{\lambda}|=5}} \frac{(\rho_{pq} + k_{pq} \bar{S}_{pq})^{2-\lambda_1} \bar{S}_{pp}^{\lambda_1} \bar{S}_{qq}^{\lambda_2} \bar{S}_{pq}^{\lambda_3}}{(1 + k_{pq} \bar{S}_{pp})^{1+\lambda_2} (1 + k_{pq} \bar{S}_{qq})^{1+\lambda_3}},$$

for some $k_{pq} = k_{pq}(S_{pp}, S_{qq}, S_{pq}) \in (0, 1)$, is the remainder in Lagrange's form. The ‘almost surely’ qualifier is in (2.4) because on an event of measure zero, either S_{pp} or S_{qq} may be zero, in which case the Taylor's theorem doesn't apply since f is defined on $\mathbb{R}_{\geq 0}^2 \times \mathbb{R}$. Our proof depends crucially on recognizing that, when $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3) = (0, 0, 2)$,

$$\begin{aligned} & \frac{\partial^{\boldsymbol{\lambda}} f(1, 1, \rho_{pq})}{\boldsymbol{\lambda}!} \bar{S}_{pp}^{\lambda_1} \bar{S}_{qq}^{\lambda_2} \bar{S}_{pq}^{\lambda_3} = \bar{S}_{pq}^2 \\ &= \frac{\sum_{i=1}^n (X_{pi} X_{qi} - \rho_{pq})^2}{n^2} + \frac{2 \sum_{1 \leq i < j \leq n} (X_{pi} X_{qi} - \rho_{pq})(X_{pj} X_{qj} - \rho_{pq})}{n^2}, \end{aligned}$$

in light of Lemma B.1 which specifies the partial derivatives of f . One can then equivalently write (2.4) as

$$(2.6) \quad \hat{\rho}_{pq}^2 - \rho_{pq}^2 = I_{pq} + II_{pq} + III_{pq},$$

where

$$(2.7) \quad I_{pq} := \frac{2 \sum_{1 \leq i < j \leq n} (X_{pi} X_{qi} - \rho_{pq})(X_{pj} X_{qj} - \rho_{pq})}{n^2}, \text{ and}$$

$$(2.8) \quad II_{pq} := \frac{\sum_{i=1}^n (X_{pi} X_{qi} - \rho_{pq})^2}{n^2} + \sum_{\substack{\boldsymbol{\lambda} \in \mathbb{N}_{\geq 0}^3: \\ 1 \leq |\boldsymbol{\lambda}| \leq 4 \\ \boldsymbol{\lambda} \neq (0, 0, 2)}} \frac{\partial^{\boldsymbol{\lambda}} f(1, 1, \rho_{pq})}{\boldsymbol{\lambda}!} \bar{S}_{pp}^{\lambda_1} \bar{S}_{qq}^{\lambda_2} \bar{S}_{pq}^{\lambda_3}.$$

Defining $I := \sum_{1 \leq p < q \leq m} I_{pq}$, $II := \sum_{1 \leq p < q \leq m} II_{pq}$ and $III := \sum_{1 \leq p < q \leq m} III_{pq}$ by summing over all $1 \leq p < q \leq m$, from (2.6) one can write

$$(2.9) \quad T - \frac{m(m-1)}{2n} - 2^{-1} \|\mathbf{R} - \mathbf{I}_m\|_F^2 = I + \left(II - \frac{m(m-1)}{2n} \right) + III,$$

realizing that $2^{-1} \|\mathbf{R} - \mathbf{I}_m\|_F^2 = \sum_{1 \leq p < q \leq m} \rho_{pq}^2$. We are now in the position to introduce three supporting lemmas that are the building blocks of Theorem 2.1.

The first lemma gives a Berry-Esseen bound for the cumulative distribution function of the term I with $\Phi(\cdot)$ after standardization. This will ultimately drive the form of our minimax power function in Theorem 2.1. The next two lemmas control the variability of the extra terms, $(II - \frac{m(m-1)}{2n})$ and III . From now on for the rest of this paper all the big O , little o notations are with respect to our considered asymptotic regime $m, n \rightarrow \infty, m/n \leq \kappa$.

Lemma 2.3 (Berry Esseen theorem for I). *The following are true for I :*

(i) *Variance:*

$$\text{Var}[I] = \mathbb{E}[I^2] = \frac{m^2}{n^2} + o\left(\frac{m^{2(1-\gamma)}}{n^2}\right) \sum_{k=0}^2 \|\mathbf{R} - \mathbf{I}_m\|_F^{2k}$$

for any $0 < \gamma < 1/2$.

(ii) *Berry-Esseen bound:*

$$\sup_{t \in \mathbb{R}} \left| P\left(\frac{I}{\sqrt{\text{Var}(I)}} \leq t\right) - \Phi(t) \right| \lesssim \left\{ \frac{o(m^4/n^4) \sum_{k=0}^8 \|\mathbf{R} - \mathbf{I}_m\|_F^k}{\text{Var}(I)^2} \right\}^{1/5}.$$

Lemma 2.4 (Bound on the 2nd moment of $II - \frac{m(m-1)}{2n}$).

$$(2.10) \quad \mathbb{E} \left[\left(II - \frac{m(m-1)}{2n} \right)^2 \right] \lesssim \frac{\|\mathbf{R} - \mathbf{I}_m\|_F^2 + \|\mathbf{R} - \mathbf{I}_m\|_F^4}{n} + o\left(\frac{m^{2(1-\gamma)}}{n^2}\right) \sum_{k=0}^4 \|\mathbf{R} - \mathbf{I}_m\|_F^k,$$

for any fixed $0 < \gamma < 1/2$.

Lemma 2.5 (Probability bound for III). *For any $0 < c < \frac{1}{2}$, there exists $C > 0$ such that*

$$P\left(|III| > C \frac{m^2}{n^{5c}}\right) \lesssim (n^{c-1} \log m + n^{c-1/2} \sqrt{\log m})$$

for large enough m, n .

The proofs of Lemmas 2.3 and 2.4 are separately given in the next two sections. Lemma 2.5 is proved by a standard maximal inequality in Appendix A. With these tools we can now establish Theorem 2.1 based on the general approach laid out in Cai and Ma (2013).

Proof of Theorem 2.1. From (2.3) and (2.9) the power of our test can be written as

$$(2.11) \quad \mathbb{E}[\psi] = P\left(I + II + III - \frac{m(m-1)}{2n} > \frac{m}{n} z_\alpha - 2^{-1} \|\mathbf{R} - \mathbf{I}_m\|_F^2\right).$$

By dividing the set $\Theta(b)$ into two subsets

$$\Theta(b, B) = \{\mathbf{R} : B\sqrt{m/n} > \|\mathbf{R} - \mathbf{I}_m\|_F \geq b\sqrt{m/n}\}$$

and

$$\Theta(B) = \{\mathbf{R} : \|\mathbf{R} - \mathbf{I}_m\|_F \geq B\sqrt{m/n}\},$$

where B is a sufficiently large constant depending on (α, b, κ) , it suffices to show

$$(2.12) \quad \liminf_{n \rightarrow \infty} \inf_{\Theta(B)} \mathbb{E}_{\mathbf{R}}[\psi] \geq \bar{\Phi}\left(z_\alpha - \frac{b^2}{2}\right)$$

and

$$(2.13) \quad \sup_{\Theta(b, B)} \left| \mathbb{E}_{\mathbf{R}}\psi - \bar{\Phi}\left(z_\alpha - \frac{\|\mathbf{R} - \mathbf{I}_m\|_F^2}{2m/n}\right) \right| \rightarrow 0$$

as $m, n \rightarrow \infty$, $m/n \leq \kappa$. Together, they lead to the theorem since (2.13) implies that

$$\lim_{n \rightarrow \infty} \inf_{\Theta(b, B)} \mathbb{E}_{\mathbf{R}}\psi = \lim_{n \rightarrow \infty} \inf_{\Theta(b, B)} \bar{\Phi}\left(z_\alpha - \frac{\|\mathbf{R} - \mathbf{I}_m\|_F^2}{2m/n}\right) = \bar{\Phi}\left(z_\alpha - \frac{b^2}{2}\right).$$

To prove (2.12) we first suppose that B is larger than $\sqrt{3z_\alpha}$, and let δ be any positive constant satisfying $0 < \delta \leq 4^{-1}z_\alpha$. By definition, for any $\mathbf{R} \in \Theta(B)$, it must be the case that $\|\mathbf{R} - \mathbf{I}_m\|_F = \tau\sqrt{m/n}$ for some $\tau \geq B$. Together with the fact that $mn^{-1}z_\alpha - 2^{-1}\|\mathbf{R} - \mathbf{I}_m\|_F^2 \leq -\frac{m\tau^2}{n6}$ and $\delta \leq 12^{-1}\tau^2$ which are consequences of the choice of B , by a union bound and Chebyshev's inequality we continue from (2.11) and obtain

$$(2.14) \quad \begin{aligned} 1 - \mathbb{E}[\psi] &\leq P\left(\left|I + II - \frac{m(m-1)}{2n}\right| \geq \frac{\tau^2 m}{6n} - \delta \frac{m}{n}\right) + P(|III| > \delta \frac{m}{n}) \\ &\leq 288\tau^{-4}n^2m^{-2} \left(\mathbb{E}[I^2] + \mathbb{E}\left[\left(II - \frac{m(m-1)}{2n}\right)^2\right] \right) + P(|III| > \delta \frac{m}{n}). \end{aligned}$$

Substituting $\|\mathbf{R} - \mathbf{I}_m\|_F$ for $\tau\sqrt{m/n}$ into the bounds for $\mathbb{E}[I^2]$ and $\mathbb{E}[(II - \frac{m(m-1)}{2n})^2]$ in Lemmas 2.3 and 2.4, it is seen that the first term in (2.14) is bounded by a term of order

$$\tau^{-4} + o(1) \left(\sum_{k=0}^4 \tau^{-k} \right)$$

Moreover, the second term in (2.14) converges to 0 as $m, n \rightarrow \infty$ by Lemma 2.5 since $\delta m/n$ is larger than m^2/n^{5c} asymptotically for any constant $2/5 < c < 1/2$, given that $m/n \leq \kappa$. They together imply that the constant $B = B(\alpha, b, \kappa)$ can be taken large enough so that

$$1 - \inf_{\Theta(B)} \mathbb{E}_{\mathbf{R}}[\psi] \leq \bar{\Phi}\left(z_\alpha - \frac{b^2}{2}\right) \text{ as } m, n \rightarrow \infty,$$

which is equivalent to (2.12).

To show (2.13), the uniform convergence of power on the “stripe” of alternatives with the signal $\|\mathbf{R} - \mathbf{I}_m\|_F$ bounded from above and below in size, we shall first establish that

$$(2.15) \quad P\left(|\tilde{I}| \geq \frac{m^{1-\gamma}}{n}\right) = o(1) \quad \text{as } m, n \rightarrow \infty \quad \text{and} \quad m/n \leq \kappa,$$

uniformly over the set $\Theta(b, B)$, where

$$\tilde{I} := II - \frac{m(m-1)}{2n} + III.$$

and γ is any number such that $0 < \gamma < 1/2$. By a union bound we have

$$(2.16) \quad \begin{aligned} P\left(|\tilde{I}| \geq \frac{m^{1-\gamma}}{n}\right) &\leq P\left(|III| \geq \frac{m^{1-\gamma}}{2n}\right) + P\left(\left|II - \frac{m(m-1)}{2n}\right| \geq \frac{m^{1-\gamma}}{2n}\right) \\ &\lesssim n^{c-1} \log m + n^{c-1/2} \sqrt{\log m} + \frac{n^2}{m^{2(1-\gamma)}} \mathbb{E} \left[\left(II - \frac{m(m-1)}{2n} \right)^2 \right] \end{aligned}$$

for any $(2 + \gamma)/5 < c < 1/2$ and large enough m, n . The last inequality comes from the Chebyshev inequality and the fact that, by taking $(2 + \gamma)/5 < c < 1/2$ in Lemma 2.5, for large enough m, n , under $m/n \leq \kappa$, we have

$$P\left(|III| \geq \frac{m^{1-\gamma}}{2n}\right) \leq P\left(|III| \geq \frac{m^2}{2\kappa^{1+\gamma}n^{2+\gamma}}\right) \leq P\left(|III| \geq C \frac{m^2}{n^{5c}}\right),$$

where the constant C is same as the one in Lemma 2.5. Since $\mathbf{R} \in \Theta(b, B)$, it must be that $\|\mathbf{R} - \mathbf{I}_m\|_F = \tau \sqrt{m/n}$ for some $b \leq \tau \leq B$, and substituting this into the variance bound in Lemma 2.4 it can be easily seen that

$$\frac{n^2}{m^{2(1-\gamma)}} \mathbb{E} \left[\left(II - \frac{m(m-1)}{2n} \right)^2 \right] \rightarrow 0$$

uniformly over $\Theta(b, B)$ as $m, n \rightarrow \infty$, $m/n \leq \kappa$. This gives (2.15) since $c < 1/2$ in (2.16).

To finish the proof of (2.13), by union bound arguments one has

$$\mathbb{E}[\psi] \leq P\left(I \geq \frac{mz_\alpha}{n} - \frac{\|\mathbf{R} - \mathbf{I}_m\|_F^2}{2} - \frac{m^{1-\gamma}}{n}\right) + P\left(|\tilde{I}| \geq \frac{m^{1-\gamma}}{n}\right)$$

and

$$\mathbb{E}[\psi] \geq P\left(I \geq \frac{mz_\alpha}{n} - \frac{\|\mathbf{R} - \mathbf{I}_m\|_F^2}{2} + \frac{m^{1-\gamma}}{n}\right) - P\left(|\tilde{I}| \geq \frac{m^{1-\gamma}}{n}\right),$$

which collectively imply

$$(2.17) \quad \begin{aligned} &\left| \mathbb{E}[\psi] - \bar{\Phi} \left(\frac{mz_\alpha n^{-1} - 2^{-1} \|\mathbf{R} - \mathbf{I}_m\|_F^2}{\sqrt{\text{Var}(I)}} \right) \right| \\ &\leq \sup_{t \in \mathbb{R}} \left| P \left(\frac{I}{\sqrt{\text{Var}(I)}} \geq t \right) - \bar{\Phi}(t) \right| + 2P\left(|\tilde{I}| \geq \frac{m^{1-\gamma}}{n}\right) + \frac{2m^{1-\gamma}n^{-1}}{\sqrt{\text{Var}(I)}} \end{aligned}$$

since $|\bar{\Phi}(x \pm \epsilon) - \bar{\Phi}(x)| \leq \epsilon$ for any $x \in \mathbb{R}$ and $\epsilon \geq 0$. Moreover, all three terms on the right hand side of (2.17) are of order $o(1)$ uniformly over $\Theta(b, B)$. The first two terms are so by Lemma 2.3(ii) and (2.15), and the last term is so since by Lemma 2.3(i), $\sqrt{\text{Var}(I)} = m/n + o(m^{1-\gamma}/n)$ where the $o(m^{1-\gamma}/n)$ term is also uniform over $\Theta(b, B)$. Finally, by Lemma 2.3(i) as $m, n \rightarrow \infty$, $m/n \leq \kappa$, we also have

$$\sup_{\Theta(b, B)} \left| \frac{\text{Var}(I)}{4m^2/n^2} - 1 \right| \rightarrow 0,$$

and it is not hard to see that this implies

$$\sup_{\Theta(b,B)} \left| \bar{\Phi} \left(z_\alpha - \frac{\|\mathbf{R} - \mathbf{I}_m\|_F^2}{2m/n} \right) - \bar{\Phi} \left(\frac{mz_\alpha n^{-1} - 2^{-1} \|\mathbf{R} - \mathbf{I}_m\|_F^2}{\sqrt{\text{Var}(I)}} \right) \right| \rightarrow 0.$$

Applying these facts to (2.17) leads to (2.13). \square

Remarks. In establishing Theorem 2.1 above, perhaps the most important step is singling out I as the main term that drives the asymptotic normality of the left hand side in (2.9) under the “stripe” of alternative $\Theta(b, B)$ via the Berry-Esseen bound in Lemma 2.3(ii). We note that I is already a rather simple term to handle, but proving Lemma 2.3(ii) for it still takes considerable effort as seen in the next section. Moreover, we didn’t intend to obtain the sharpest possible bounds in our three building lemmas: Lemmas 2.3, 2.4 and 2.5; they are nevertheless good enough for our purpose. Lastly, our Theorem 2.1 is slightly weaker than the parallel result of Cai and Ma (2013, Theorem 4) in that an upper bound on the ratio m/n is imposed. This is mainly because in our proof in order to show that the remainder term III in (2.9) tends to zero in probability, we applied the tail bound in Lemma 2.5, a crude estimate based on a maximal inequality shown in Appendix A. At this moment, we cannot think of other ways to control this term.

3. THE BERRY ESSEEN BOUND FOR I

We will prove Lemma 2.3 in this section. For our presentation, given a finite set D and $|D|$ duples $(p_d, q_d) \in [m] \times [m]$ indexed by a subscript d that ranges over D , we define the central moment quantities

$$\mathcal{M}_{(p_d, q_d)} := \mathbb{E} \left[\prod_{d \in D} (X_{p_d} X_{q_d} - \rho_{p_d q_d}) \right].$$

Recall that I is defined as $\sum_{p < q} I_{pq}$, where each I_{pq} is given in (2.7). We first observe that I has a natural martingale structure: For each $i = 1, \dots, n$, let \mathcal{F}_i be the sigma-algebra generated by $\{X_{pj} : 1 \leq p \leq m; 1 \leq j \leq i\}$ and \mathcal{F}_0 be the trivial sigma algebra, and define

$$(3.1) \quad Y_i := \frac{2}{n^2} \sum_{p < q} \sum_{j < i} (X_{pi} X_{qi} - \rho_{pq})(X_{pj} X_{qj} - \rho_{pq}) \text{ for } i = 2, \dots, n$$

as well as

$$(3.2) \quad Y_0 = Y_1 := 0.$$

Then $I = \sum_{i=0}^n Y_i$, and $(Y_i)_{i=0}^n$ is a the sequence of martingale differences since

$$\mathbb{E}[Y_i | \mathcal{F}_{i-1}] = \sum_{p < q} \frac{2}{n^2} \sum_{j < i} (X_{pj} X_{qj} - \rho_{pq}) \mathbb{E}[X_{pi} X_{qi} - \rho_{pq}] = 0$$

for $i \geq 2$, where $\mathbb{E}[Y_i | \mathcal{F}_{i-1}] = 0$ is trivial for $i = 0, 1$.

With the observations just made it is easy to see that $\mathbb{E}[I] = 0$ and

$$(3.3) \quad \text{Var}[I] = \mathbb{E}[I^2] = \sum_{i=2}^n \mathbb{E}[Y_i^2].$$

By the i.i.d.'ness of the samples, for each $i = 2, \dots, n$,

$$(3.4) \quad \begin{aligned} \mathbb{E}[Y_i^2] &= \frac{4}{n^4} \sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2}} \mathcal{M}_{(p_d, q_d)} \left(\sum_{1 \leq j, j' < i} \mathbb{E}[(X_{p_1 j'} X_{q_1 j'} - \rho_{p_1 q_1})(X_{p_2 j} X_{q_2 j} - \rho_{p_2 q_2})] \right) \\ &= \frac{4(i-1)}{n^4} \sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2}} \mathcal{M}_{(p_d, q_d)}^2, \end{aligned}$$

where, to clarify, $\sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2}}$ means a summation over all pairs of duples $\{(p_1, q_1), (p_2, q_2)\}$ such that $1 \leq p_d < q_d \leq m$ for each $d = 1, 2$. We have the equality in (3.4) because $\mathbb{E}[(X_{p_1 j'} X_{q_1 j'} - \rho_{p_1 q_1})(X_{p_2 j} X_{q_2 j} - \rho_{p_2 q_2})]$ equals $\mathcal{M}_{(p_d, q_d)}$ when $j = j'$ and zero otherwise. For $k = 2, 3, 4$, let

$$(3.5) \quad \mathbb{S}(k) := \sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2 \\ |\cup_{d=1}^2 \{p_d, q_d\}|=k}} \mathcal{M}_{(p_d, q_d)}^2$$

correspond to a sum over all duples $1 \leq p_d < q_d \leq m$, $d = 1, 2$ such that as a set $\cup_{d=1}^2 \{p_d, q_d\}$ has cardinality k . From (3.3) and (3.4) we can write

$$(3.6) \quad \text{Var}[I] = \frac{2n(n-1)}{n^4} \sum_{k=2}^4 \mathbb{S}(k).$$

since $\sum_{i=2}^n (i-1) = 2^{-1}(n^2 - n)$. In Appendix C, we will show the following estimates hold:

$$(3.7) \quad \mathbb{S}(2) = 2^{-1}m(m-1) + O(\|\mathbf{R} - \mathbf{I}_m\|_F^2)$$

$$(3.8) \quad \mathbb{S}(3) = O(m\|\mathbf{R} - \mathbf{I}_m\|_F^2 + \|\mathbf{R} - \mathbf{I}_m\|_F^4)$$

$$(3.9) \quad \mathbb{S}(4) = O(\|\mathbf{R} - \mathbf{I}_m\|_F^4)$$

Substituting these into (3.6) results in Lemma 2.3(i). In fact, this general strategy of decomposing a sum according to the cardinality of an index set as in (3.5) and forming separate estimates will be employed repeatedly in the sequel.

We shall now prove the normal approximation in Lemma 2.3(ii). With a Berry-Esseen theorem for martingale central limit theorem in Heyde and Brown (1970), it suffices to verify the fourth moment conditions

$$(3.10) \quad \sum_{i=2}^n \mathbb{E}[Y_i^4] = o(m^4/n^4) \sum_{k=0}^4 \|\mathbf{R} - \mathbf{I}_m\|_F^k$$

and

$$\begin{aligned}
\mathbb{E} \left[\left(\sum_{i=2}^n \mathbb{E}[Y_i^2 | \mathcal{F}_{i-1}] - \text{Var}(I) \right)^2 \right] &= \mathbb{E} \left[\left(\sum_{i=2}^n \mathbb{E}[Y_i^2 | \mathcal{F}_{i-1}] \right)^2 \right] - \text{Var}(I)^2 \\
(3.11) \qquad \qquad \qquad &= o(m^4/n^4) \sum_{k=0}^8 \|\mathbf{R} - \mathbf{I}_m\|_F^k.
\end{aligned}$$

Note that the equality before (3.11) holds because $\mathbb{E}[\sum_{i=2}^n \mathbb{E}[Y_i^2 | \mathcal{F}_{i-1}]] = \mathbb{E}[\sum_{i=2}^n Y_i^2] = \text{Var}(I)$.

We will first show (3.10). For any $2 \leq i \leq n$, on raising Y_i to the 4th power and taking expectation, by the i.i.d.'ness of samples, we have

$$\begin{aligned}
&\mathbb{E}[Y_i^4] \\
&= \frac{16}{n^8} \sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2,3,4}} \left\{ \mathbb{E} \left[\prod_{d=1}^4 (X_{p_d i} X_{q_d i} - \rho_{p_d q_d}) \right] \sum_{\substack{1 \leq j_d < i \\ d=1,2,3,4}} \mathbb{E} \left[\prod_{d=1}^4 (X_{p_d j_d} X_{q_d j_d} - \rho_{p_d q_d}) \right] \right\} \\
&= \frac{16}{n^8} \sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2,3,4}} \left\{ \mathcal{M}_{(p_d, q_d)} \sum_{\substack{d \in [4] \\ 1 \leq j_d < i \\ d=1,2,3,4}} \mathbb{E} \left[\prod_{d=1}^4 (X_{p_d j_d} X_{q_d j_d} - \rho_{p_d q_d}) \right] \right\} \\
(3.12) \qquad \qquad \qquad &= O\left(\frac{i^2}{n^8}\right) \sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2,3,4}} \mathcal{M}_{(p_d, q_d)},
\end{aligned}$$

where the summations $\sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2,3,4}}$ and $\sum_{\substack{1 \leq j_d < i \\ d=1,2,3,4}}$ are defined similarly as the one in (3.4). The last equality in (3.12) is explained as follows: For a fixed i and a given set of variables index pairs $\{(p_d, q_d) : d = 1, \dots, 4\}$, with any choice of the sample indices j_1, \dots, j_4 in order for the expectation

$$(3.13) \qquad \qquad \qquad \mathbb{E} \left[\prod_{d=1}^4 (X_{p_d j_d} X_{q_d j_d} - \rho_{p_d q_d}) \right]$$

to be non-zero, by independence it must be true that there exists a permutation function $\pi \in \mathcal{S}_4$ so that

$$(3.14) \qquad \qquad \qquad j_{\pi(1)} = j_{\pi(2)}, \quad j_{\pi(3)} = j_{\pi(4)}.$$

Since the condition in (3.14) implies that $|\cup_{d=1}^4 \{j_d\}| \leq 2$, at most $O(\binom{i-1}{2}) = O(i^2)$ many expectations in (3.13) can be non-zero. This leads to (3.12) since the expectations in (3.13), when they are non-zero, can be uniformly bounded regardless of the choice for $\{(p_d, q_d, j_d); d = 1, \dots, 4\}$, owing to our assumptions at the beginning of Section 2 and Theorem B.2 on higher order normal moments.

Provided that $\sum_{i=2}^n i^2 = 6^{-1}(2n^3 + 3n^2 + n - 6)$, with (3.12) we further write

$$(3.15) \quad \sum_{i=2}^n \mathbb{E}[Y_i^4] = O(n^{-5}) \sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2,3,4}} \mathcal{M}_{(p_d, q_d), d \in [4]}.$$

Now the last term in (3.15) can be decomposed, according to the cardinality of the set of duples $\cup_{d=1}^4 \{p_d, q_d\}$, as

$$(3.16) \quad \sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2,3,4}} \mathcal{M}_{(p_d, q_d), d \in [4]} = \sum_{k=5}^8 \mathbb{T}(k) + O(m^4),$$

where for $k = 2, \dots, 8$,

$$\mathbb{T}(k) := \sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2,3,4 \\ |\cup_{d=1}^4 \{p_d, q_d\}| = k}} \mathcal{M}_{(p_d, q_d), d \in [4]}$$

and the $O(m^4)$ term comes from the fact that there are only $O(m^4)$ many uniformly bounded extra summands under the restriction $|\cup_{k=1}^4 \{p_d, q_d\}| \leq 4$. In Appendix C we will show that

$$(3.17) \quad \mathbb{T}(k) = O(m^4) \|\mathbf{R} - \mathbf{I}_m\|_F^{k-4}$$

for each $k = 5, \dots, 8$. Collecting (3.15), (3.16) and (3.17) we get (3.10).

To show (3.11) it suffices to understand the term $\mathbb{E}[(\sum_{i=1}^n \mathbb{E}[Y_i^2 | \mathcal{F}_{i-1}])^2]$ since the form of $\text{Var}(I)$ has been proven in Lemma 2.3(i). On expansion,

$$(3.18) \quad \sum_{i=2}^n \mathbb{E}[Y_i^2 | \mathcal{F}_{i-1}] = \frac{4}{n^4} \sum_{i=2}^n \sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2}} \mathcal{M}_{(p_d, q_d), d \in [2]} \left[\sum_{1 \leq j, k < i} (X_{p_1 j} X_{q_1 j} - \rho_{p_1 q_1})(X_{p_2 k} X_{q_2 k} - \rho_{p_2 q_2}) \right].$$

Proceeding with our calculations,

$$(3.19) \quad \mathbb{E} \left[\left(\sum_{i=2}^n \mathbb{E}[Y_i^2 | \mathcal{F}_{i-1}] \right)^2 \right] = \frac{16}{n^8} \sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2,3,4}} \left\{ \mathbb{P}_1 \times \sum_{2 \leq i, j \leq n} \sum_{\substack{1 \leq i_1, i_2 < i \\ 1 \leq i_3, i_4 < j}} \mathbb{E} \left[\prod_{d=1}^4 (X_{p_d i_d} X_{q_d i_d} - \rho_{p_d q_d}) \right] \right\},$$

where

$$(3.20) \quad \mathbb{P}_1 = \mathbb{P}_1(p_1, q_1, \dots, p_4, q_4) := \mathcal{M}_{(p_d, q_d), d \in \{1,2\}} \mathcal{M}_{(p_d, q_d), d \in \{3,4\}}$$

By independence, we note that the expression

$$\mathbb{E} \left[\prod_{d=1}^4 (X_{p_d i_d} X_{q_d i_d} - \rho_{p_d q_d}) \right]$$

on the right hand side of (3.19) can be non-zero only if the four sample indices i_1, \dots, i_4 are such that either

$$(3.21) \quad i_1 = \dots = i_4,$$

$$(3.22) \quad i_1 = i_2, \quad i_3 = i_4, \quad |\{i_1, \dots, i_4\}| = 2,$$

$$(3.23) \quad i_1 = i_3, \quad i_2 = i_4, \quad |\{i_1, \dots, i_4\}| = 2$$

or

$$(3.24) \quad i_1 = i_4, \quad i_2 = i_3, \quad |\{i_1, \dots, i_4\}| = 2.$$

For any fixed given pair $2 \leq i, j \leq n$, by simple counting, there are, respectively, $i \wedge j - 1$, $(i \wedge j - 1)(i \vee j - 2)$, $(i \wedge j - 1)(i \wedge j - 2)$, $(i \wedge j - 1)(i \wedge j - 2)$ combinations of (i_1, i_2, i_3, i_4) that satisfy (3.21), (3.22), (3.23), (3.24) for which $1 \leq i_1, i_2 < i$ and $1 \leq i_3, i_4 < j$, where $a \vee b = \max(a, b)$ and $a \wedge b = \min(a, b)$. Hence,

$$\begin{aligned} & \sum_{2 \leq i, j \leq n} \sum_{\substack{1 \leq i_1, i_2 < i \\ 1 \leq i_3, i_4 < j}} \mathbb{E} \left[\prod_{d=1}^4 (X_{p_d i_d} X_{q_d i_d} - \rho_{p_d q_d}) \right] \\ &= \mathcal{M}_{(p_d, q_d)} \underbrace{\left\{ \sum_{2 \leq i, j \leq n} (i \wedge j - 1) \right\}}_{=6^{-1}(2n^3 - 3n^2 + n)} + \mathbb{P}_1 \underbrace{\sum_{2 \leq i, j \leq n} (i \wedge j - 1)(i \vee j - 2)}_{=12^{-1}(-2n + 9n^2 - 10n^3 + 3n^4)} \\ & \quad + (\mathbb{P}_2 + \mathbb{P}_3) \underbrace{\left\{ \sum_{2 \leq i, j \leq n} (i \wedge j - 1)(i \wedge j - 2) \right\}}_{=6^{-1}(n^4 - 4n^3 + 5n^2 - 2n)} \\ (3.25) \quad &= \mathcal{M}_{(p_d, q_d)} O(n^3) + \mathbb{P}_1 \left(\frac{n^4}{4} + O(n^3) \right) + (\mathbb{P}_2 + \mathbb{P}_3) O(n^4), \end{aligned}$$

where

$$\begin{aligned} \mathbb{P}_2 &= \mathbb{P}_2(p_1, q_1, \dots, p_4, q_4) := \mathcal{M}_{(p_d, q_d)} \mathcal{M}_{(p_d, q_d)} \\ & \quad d \in \{1, 3\} \quad d \in \{2, 4\} \\ \mathbb{P}_3 &= \mathbb{P}_3(p_1, q_1, \dots, p_4, q_4) := \mathcal{M}_{(p_d, q_d)} \mathcal{M}_{(p_d, q_d)} \\ & \quad d \in \{1, 4\} \quad d \in \{2, 3\} \end{aligned}$$

are the value of $\mathbb{E}[\prod_{d=1}^4 (X_{p_d i_d} X_{q_d i_d} - \rho_{p_d q_d})]$ when i_1, \dots, i_4 satisfy the criteria (3.23) and (3.24) respectively. Substituting (3.25) into (3.19) gives

$$(3.26) \quad \mathbb{E} \left[\left(\sum_{i=2}^n \mathbb{E}[Y_i^2 | \mathcal{F}_{i-1}] \right)^2 \right] =$$

$$O(n^{-5}) \sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2,3,4}} \mathbb{P}_1 + \left(\frac{4}{n^4} + O(n^{-5}) \right) \sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2,3,4}} \mathbb{P}_1^2 + O(n^{-4}) \sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2,3,4}} \sum_{u=2}^3 \mathbb{P}_1 \mathbb{P}_u,$$

where the terms $\mathcal{M}_{(p_d, q_d)}^{d \in [4]}$ in (3.25) are absorbed into the first $O(n^{-5})$ term because they are uniformly bounded regardless of the choice of $p_1, q_1, \dots, p_4, q_4$, again by our assumptions and Theorem B.2. From this it remains to show the estimates

$$(3.27) \quad \sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2,3,4}} \mathbb{P}_1 = O(m^4) \sum_{k=0}^4 \|\mathbf{R} - \mathbf{I}_m\|_F^k,$$

$$(3.28) \quad \sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2,3,4}} \mathbb{P}_1^2 = \frac{m^4}{4} + O(m^3) \sum_{k=0}^8 \|\mathbf{R} - \mathbf{I}_m\|_F^k,$$

and

$$(3.29) \quad \sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2,3,4}} \sum_{u=2}^3 \mathbb{P}_1 \mathbb{P}_u = O(m^3) \sum_{k=0}^8 \|\mathbf{R} - \mathbf{I}_m\|_F^k,$$

which, together with Lemma 2.3(i) and (3.26), imply (3.11). The proofs of these estimates will, again, be deferred to Appendix C.

4. THE SECOND MOMENT BOUND FOR $II - \frac{m(m-1)}{2n}$

We will now prove Lemma 2.4. Recall that $II := \sum_{p < q} II_{pq}$, and from the definition of II_{pq} in (2.8) we can equivalently write it as

$$II_{pq} = II_{pq,1} + II_{pq,2},$$

where

$$(4.1) \quad II_{pq,1} := \frac{\sum_{i=1}^n (X_{pi} X_{qi} - \rho_{pq})^2}{n^2} + \sum_{\substack{\boldsymbol{\lambda} \in \mathbb{N}_{\geq 0}^3: \\ 3 \leq |\boldsymbol{\lambda}| \leq 4 \\ \lambda_3 = 2 \\ \boldsymbol{\lambda} \neq (1,1,2)}} \frac{\partial^{\boldsymbol{\lambda}} f(1,1,\rho_{pq})}{\boldsymbol{\lambda}!} \bar{S}_{pp}^{\lambda_1} \bar{S}_{qq}^{\lambda_2} \bar{S}_{pq}^{\lambda_3}$$

and

$$(4.2) \quad II_{pq,2} := \frac{\partial^{(1,1,2)} f(1,1,\rho_{pq})}{1!1!2!} \bar{S}_{pp} \bar{S}_{qq} \bar{S}_{pq}^2 + \sum_{\substack{\boldsymbol{\lambda} \in \mathbb{N}_{\geq 0}^3: \\ 1 \leq |\boldsymbol{\lambda}| \leq 4 \\ \lambda_3 \neq 2}} \frac{\partial^{\boldsymbol{\lambda}} f(1,1,\rho_{pq})}{\boldsymbol{\lambda}!} \bar{S}_{pp}^{\lambda_1} \bar{S}_{qq}^{\lambda_2} \bar{S}_{pq}^{\lambda_3}.$$

We form this grouping of terms for reasons that will be explained later. As such, by defining $II_1 := \sum_{p < q} II_{pq,1}$ and $II_2 := \sum_{p < q} II_{pq,2}$, one can write

$$II = II_1 + II_2.$$

To finish the proof of Lemma 2.4, it suffices to bound the second moments of $II_1 - \frac{m(m-1)}{2n}$ and II_2 respectively in terms of $\|\mathbf{R} - \mathbf{I}_m\|_F$.

Lemma 4.1 (Bound on the second moment of $II_1 - \frac{m(m-1)}{2n}$).

$$\mathbb{E} \left[\left(II_1 - \frac{m(m-1)}{2n} \right)^2 \right] \lesssim o \left(\frac{m^{2(1-\gamma)}}{n^2} \right) \sum_{k=0}^4 \|\mathbf{R} - \mathbf{I}\|_F^k$$

for any $0 < \gamma < 1/2$.

Lemma 4.2 (Bound on the second moment of II_2).

$$(4.3) \quad \mathbb{E} \left[(II_2)^2 \right] \lesssim \frac{\|\mathbf{R} - \mathbf{I}_m\|_F^2 + \|\mathbf{R} - \mathbf{I}_m\|_F^4}{n} + o \left(\frac{m^{2(1-\gamma)}}{n^2} \right) \sum_{k=0}^2 \|\mathbf{R} - \mathbf{I}\|_F^k$$

for any $0 < \gamma < 1/2$.

Using Lemmas 4.1 and 4.2, Lemma 2.4 immediately follows from (i) $II^2 = (II_1 - \frac{m(m-1)}{2})^2 + II_2^2 + 2(II_1 - \frac{m(m-1)}{2})II_2$ and (ii) $2|(II_1 - \frac{m(m-1)}{2})II_2| \leq (II_1 - \frac{m(m-1)}{2})^2 + II_2^2$.

For each pair $p < q$, the main difference between $II_{pq,1}$ and $II_{pq,2}$ is that when $\lambda_3 \neq 2$, all the coefficients $\frac{\partial^\lambda f(1,1,\rho_{pq})}{\lambda!}$ appearing in the second term of (4.2) can be bounded by either $|\rho_{pq}|$ or ρ_{pq}^2 up to some multiplicative constants. This makes proving the useful bound for $\mathbb{E}[II_2^2]$ in terms of the norm $\|\mathbf{R} - \mathbf{I}_m\|_F$ amenable to the straightforward approach of squaring and taking expectation. Thus we shall defer the proof of Lemma 4.2 to Appendix D and address the bound in Lemma 4.1 for the rest of this section.

We will start with the fact that

$$(4.4) \quad \mathbb{E} \left[\left(II_1 - \frac{m(m-1)}{2n} \right)^2 \right] \leq 2 \left\{ \text{Var}[II_1] + \left(\mathbb{E}[II_1] - \frac{m(m-1)}{2n} \right)^2 \right\}$$

and form estimates for the terms on the right hand side. To understand the mean and variance of II_1 , it is more instructive to first recognize that each term in (4.1) can be written as a U-statistic of degree 4. For instance, for any four distinct indices $1 \leq i, j, k, l \leq n$, if we only treat $\mathbf{X}_{pq,i} = (X_{pi}, X_{qi})', \dots, \mathbf{X}_{pq,l} = (X_{pl}, X_{ql})'$ as a four tuple in \mathbb{R}^2 , the function

$$(4.5) \quad h_{1,pq}(\mathbf{X}_{pq,i}, \mathbf{X}_{pq,j}, \mathbf{X}_{pq,k}, \mathbf{X}_{pq,l}) := \frac{\binom{n}{4}}{n^2 \binom{n-1}{3}} \sum_{i' \in \{i,j,k,l\}} \{ (X_{pi'} X_{qi'} - \rho_{pq})^2 \},$$

is symmetric in its four arguments, and the first term in (4.1) can be written as the U-statistic

$$(4.6) \quad n^{-2} \sum_{i=1}^n (X_{pi} X_{qi} - \rho_{pq})^2 = \binom{n}{4}^{-1} \sum h_{1,pq}(\mathbf{X}_{pq,i}, \mathbf{X}_{pq,j}, \mathbf{X}_{pq,k}, \mathbf{X}_{pq,l})$$

where the summation on the right hand side is over all distinct unordered quadruples i, j, k, l that can be formed from $[n]$. We note that the factor $\binom{n-1}{3}$ appears as a denominator in (4.5) because for each $i \in \{1, \dots, n\}$, the summand $(X_{pi}X_{qi} - \rho_{pq})^2$ will appear only once on the left hand side of (4.6), while by the definition of $h_{1,pq}$ it will appear in $\binom{n-1}{3}$ kernels that are summed over on the right hand side of (4.6) (Since for each i , there will be $\binom{n-1}{3}$ choices of j, k, l to form a quadruple (i, j, k, l) from $\{1, \dots, n\}$). Thus, the factor $\binom{n-1}{3}$ appears as a denominator in definition (4.5) to account for the multiple counting.

Note that the other terms of the form $\frac{\partial^\lambda f(1,1,\rho_{pq})}{\lambda!} \bar{S}_{pp}^{\lambda_1} \bar{S}_{qq}^{\lambda_2} \bar{S}_{pq}^{\lambda_3}$ in (4.1) are indexed by λ equal to $(1, 0, 2)$, $(0, 1, 2)$, $(2, 0, 2)$, $(0, 2, 2)$. These terms can be represented as U-statistics of degree 4 using a similar strategy: With four *distinct* indices i, j, k, l from $[n]$, by defining the symmetric kernel function

$$\begin{aligned}
 & h_{2,pq}(\mathbf{X}_{pq,i}, \mathbf{X}_{pq,j}, \mathbf{X}_{pq,k}, \mathbf{X}_{pq,l}) \\
 &:= \underbrace{\binom{n}{4} \frac{n^{-3}}{\binom{n-3}{1}}}_{O(1)} \sum_{\substack{\{i',j',k'\} \\ \subset \{i,j,k,l\} \\ i',j',k' \text{ distinct} \\ \text{and unordered}}} \sum_{\pi \in \mathcal{S}_3} \left\{ (X_{p\pi(i')}^2 - 1)(X_{p\pi(j')}X_{q\pi(j')} - \rho_{pq})(X_{p\pi(k')}X_{q\pi(k')} - \rho_{pq}) \right\} \\
 &+ \underbrace{\binom{n}{4} \frac{n^{-3}}{\binom{n-2}{2}}}_{O(n^{-1})} \sum_{\substack{\{i',j'\} \\ \subset \{i,j,k,l\} \\ i',j' \text{ distinct} \\ \text{and unordered}}} \sum_{\pi \in \mathcal{S}_2} \left\{ (X_{p\pi(i')}^2 - 1)(X_{p\pi(j')}X_{q\pi(j')} - \rho_{pq})^2 \right. \\
 &\quad \left. + 2(X_{p\pi(i')}^2 - 1)(X_{p\pi(i')}X_{q\pi(i')} - \rho_{pq})(X_{p\pi(j')}X_{q\pi(j')} - \rho_{pq}) \right\} \\
 &+ \underbrace{\binom{n}{4} \frac{n^{-3}}{\binom{n-1}{3}}}_{O(n^{-2})} \sum_{i' \in \{i,j,k,l\}} \left\{ (X_{pi'}^2 - 1)(X_{pi'}X_{qi'} - \rho_{pq})^2 \right\},
 \end{aligned}$$

for $\lambda = (1, 0, 2)$, where above we interpret π as permutation functions on distinct elements, we have the U-statistic representation of degree 4

$$\begin{aligned}
 & \frac{\partial^{(1,0,2)} f(1,1,\rho_{pq})}{(1,0,2)!} \bar{S}_{pp} \bar{S}_{pq}^2 \\
 (4.8) \quad &= -n^{-3} \sum_{\tilde{i}, \tilde{j}, \tilde{k}=1}^n (X_{p\tilde{i}}^2 - 1)(X_{p\tilde{j}}X_{q\tilde{j}} - \rho_{pq})(X_{p\tilde{k}}X_{q\tilde{k}} - \rho_{pq}) \\
 &= -\binom{n}{4}^{-1} \sum_{\substack{\text{unordered} \\ \& \text{ distinct} \\ i,j,k,l \\ \text{from } [n]}} h_{2,pq}(\mathbf{X}_{pq,i}, \mathbf{X}_{pq,j}, \mathbf{X}_{pq,k}, \mathbf{X}_{pq,l}).
 \end{aligned}$$

Note that (4.8) simply comes from Lemma B.1. What we have done here is that, for each term $(X_{pi}^2 - 1)(X_{pj}X_{qj} - \rho_{pq})(X_{pk}X_{ql} - \rho_{pq})$ in (4.8) with $\tilde{i}, \tilde{j}, \tilde{k}$ not necessarily distinct, we find any 4 *distinct* indices i, j, k, l that contain $\tilde{i}, \tilde{j}, \tilde{k}$ as sets, and arrange the term into one of the three summands of order $O(1)$, $O(n^{-1})$ and $O(n^{-2})$ in (4.7) according to the actual set cardinality $|\{\tilde{i}, \tilde{j}, \tilde{k}\}|$, which can be equal to 1, 2 or 3. Since there are $\binom{n-|\{\tilde{i}, \tilde{j}, \tilde{k}\}|}{4-|\{\tilde{i}, \tilde{j}, \tilde{k}\}|}$ choices of distinct i, j, k, l that contain $\{\tilde{i}, \tilde{j}, \tilde{k}\}$ as sets, to account for the duplications we put the factors $\binom{n-3}{1}$, $\binom{n-2}{2}$, $\binom{n-1}{3}$ as denominators for the three summands in the definition (4.7) of the kernel. By a simple symmetry argument if we define the kernel

$$(4.9) \quad \bar{h}_{2,pq}(\mathbf{X}_{pq,i}, \mathbf{X}_{pq,j}, \mathbf{X}_{pq,k}, \mathbf{X}_{pq,l}) := h_{2,pq}(\bar{\mathbf{X}}_{pq,i}, \bar{\mathbf{X}}_{pq,j}, \bar{\mathbf{X}}_{pq,k}, \bar{\mathbf{X}}_{pq,l})$$

where $\bar{\mathbf{X}}_{pq,i} := (X_{qi}, X_{pi})'$, we have

$$\begin{aligned} & \frac{\partial^{(0,1,2)} f(1, 1, \rho_{pq})}{(0, 1, 2)!} \bar{S}_{qq} \bar{S}_{pq}^2 \\ &= - \binom{n}{4}^{-1} \sum_{\substack{\text{unordered} \\ \& \text{ distinct} \\ i, j, k, l \\ \text{from } [n]}} \bar{h}_{2,pq}(\mathbf{X}_{pq,i}, \mathbf{X}_{pq,j}, \mathbf{X}_{pq,k}, \mathbf{X}_{pq,l}). \end{aligned}$$

In the same vein, for λ equals $(2, 0, 2)$ or $(0, 2, 2)$ and four *distinct* indices i, j, k, l from $[n]$, we leave it to the reader to check that one can define a symmetric kernel $h_{3,pq}$ of degree 4 as shown in Appendix D such that

$$\frac{\partial^{(2,0,2)} f(1, 1, \rho_{pq})}{(2, 0, 2)!} \bar{S}_{pp}^2 \bar{S}_{pq}^2 = \binom{n}{4}^{-1} h_{3,pq}(\mathbf{X}_{pq,i}, \mathbf{X}_{pq,j}, \mathbf{X}_{pq,k}, \mathbf{X}_{pq,l})$$

and

$$\frac{\partial^{(0,2,2)} f(1, 1, \rho_{pq})}{(0, 2, 2)!} \bar{S}_{qq}^2 \bar{S}_{pq}^2 = \binom{n}{4}^{-1} \bar{h}_{3,pq}(\mathbf{X}_{pq,i}, \mathbf{X}_{pq,j}, \mathbf{X}_{pq,k}, \mathbf{X}_{pq,l}),$$

where

$$(4.10) \quad \bar{h}_{3,pq}(\mathbf{X}_{pq,i}, \mathbf{X}_{pq,j}, \mathbf{X}_{pq,k}, \mathbf{X}_{pq,l}) := h_{3,pq}(\bar{\mathbf{X}}_{pq,i}, \bar{\mathbf{X}}_{pq,j}, \bar{\mathbf{X}}_{pq,k}, \bar{\mathbf{X}}_{pq,l}).$$

Letting $\mathbf{X}_i = (X_{1i}, \dots, X_{mi})'$ denote the entire i -th sample, we have the degree-4 U-statistic representation for II_1 :

$$(4.11) \quad II_1 = \binom{n}{4}^{-1} \sum_{1 \leq i < j < k < l \leq n} h(\mathbf{X}_i, \mathbf{X}_j, \mathbf{X}_k, \mathbf{X}_l),$$

where

$$(4.12) \quad h(\mathbf{X}_i, \mathbf{X}_j, \mathbf{X}_k, \mathbf{X}_l) := \sum_{1 \leq p < q \leq m} (h_{1,pq} - h_{2,pq} - \bar{h}_{2,pq} + h_{3,pq} + \bar{h}_{3,pq})(\mathbf{X}_{pq,i}, \mathbf{X}_{pq,j}, \mathbf{X}_{pq,k}, \mathbf{X}_{pq,l}).$$

Hence,

$$\mathbb{E}[II_1] = \sum_{1 \leq p < q \leq m} \mathbb{E}[(h_{1,pq} - h_{2,pq} - \bar{h}_{2,pq} + h_{3,pq} + \bar{h}_{3,pq})(\mathbf{X}_{pq,1}, \mathbf{X}_{pq,2}, \mathbf{X}_{pq,3}, \mathbf{X}_{pq,4})].$$

The expectation for each of $h_{1,pq}(\cdot)$, $h_{2,pq}(\cdot)$, $h_{3,pq}(\cdot)$ in the preceding display can be computed by taking expectation for each of the product terms appearing in $\{\cdot\}$ in definitions (4.5), (4.7) as well as the counterparts in the definition of $h_{3,pq}$ in Appendix D (Note that quite a few of these expectations are simply zero due to independence of samples). Exploiting symmetry the same can be done for (4.9) and (4.10). In principle, these higher-order normal moments can all be obtained by repeatedly applying Isserlis's theorem (Theorem B.2) laboriously. With symbolic computational softwares such as `mathematica` they can however be effortlessly computed. These computations lead to

$$(4.13) \quad \mathbb{E}[II_1] = \sum_{1 \leq p < q \leq m} \frac{16 + n^2 + (80 + 8n + n^2)\rho_{pq}^2}{n^3} \\ = \frac{m(m-1)}{2n} + O(n^{-1})\|\mathbf{R} - \mathbf{I}_m\|_F^2 + O(m^2/n^3)$$

and further details are given in Appendix D. As a direct consequence of Hoeffding (1948)'s classical result on the variance of U-statistics, we also have the bound

$$(4.14) \quad \text{Var}[II_1] \lesssim \sum_{c=1}^4 n^{-c} \zeta_c,$$

where

$$\zeta_c := \mathbb{E}[g_c(\mathbf{X}_1, \dots, \mathbf{X}_c)^2]$$

and the functions $g_c : (\mathbb{R}^m)^c \rightarrow \mathbb{R}$, $c = 1, \dots, 4$, are defined as

$$(4.15) \quad g_c(x_1, \dots, x_c) := \mathbb{E}[h(\mathbf{X}_1, \dots, \mathbf{X}_4) | \mathbf{X}_1 = x_1, \dots, \mathbf{X}_c = x_c] - \mathbb{E}[h(\mathbf{X}_1, \dots, \mathbf{X}_4)].$$

Hence, forming estimates of the quantities ζ_1, \dots, ζ_4 can lead to an estimate of $\text{Var}[II_1]$.

Lemma 4.3 (Bound for the ζ_c 's).

$$(4.16) \quad \zeta_1 \lesssim \frac{\|\mathbf{R} - \mathbf{I}_m\|_F^4 + m^2(1 + \|\mathbf{R} - \mathbf{I}_m\|_F^2)}{n^2} + \frac{m^4}{n^4}$$

$$(4.17) \quad \zeta_2 \lesssim \frac{m^3(1 + \|\mathbf{R} - \mathbf{I}_m\|_F)}{n^2} + \frac{m^4}{n^4}$$

$$(4.18) \quad \zeta_3 \lesssim \|\mathbf{R} - \mathbf{I}_m\|_F^4 + m^2(1 + \|\mathbf{R} - \mathbf{I}_m\|_F^2) + \frac{m^4}{n^2}$$

$$(4.19) \quad \zeta_4 \lesssim m^3(\|\mathbf{R} - \mathbf{I}_m\|_F + 1) + \frac{m^4}{n^2}$$

Again, proving these estimates involves repeatedly applying Theorem B.2 with the help of `mathematica` and the details will be deferred to Appendix D. We note

that these estimates are by no means sharp, but suffice for our purpose. Putting Lemma 4.3 and (4.14) together, it is a routine task to check that

$$\text{Var}[II_1] \lesssim o\left(\frac{m^{2(1-\gamma)}}{n^2}\right) \sum_{k=0}^4 \|\mathbf{R} - I\|_F^k$$

for any $0 < \gamma < 1/2$. This, together with (4.4) and (4.13), proved Lemma 4.1.

5. CONCLUSION

In this paper, we studied the minimax power of the Rao's score statistic for testing independence, under the asymptotic regime where both the dimension m and sample size n grow to infinity when the ratio m/n is bounded. Our main result suggested that the Rao's score test is rate optimal under this regime. As an ending remark we note that while previous related work (Chen and Shao, 2012) on the null theory only requires the random variables to have finite moments, our power analysis heavily relied on the normality assumption. It is a challenge to establish power results along the same line under more relaxed conditions.

APPENDIX A. PROBABILITY TAIL BOUND OF III

We will prove the tail bound for III in Lemma 2.5. For $1 \leq p, q \leq m$, by a standard trick (Bickel and Levina, 2008, p.221), for any $t > 0$, one can show the sub-exponential inequality

$$P(|\bar{S}_{pq}| > t) \leq 4 \exp\left(\frac{-t^2}{n^{-1}2(1 + \rho_{pq})(2(1 + \rho_{pq}) + t)}\right)$$

under our assumptions at the beginning of Section 2. Then by the maximal inequality in van der Vaart and Wellner (1996, Lemma 2.2.10) and a union bound, we have for any $0 < c < 1/2$,

$$(A.1) \quad P\left(\max_{1 \leq p, q \leq m} |\bar{S}_{pq}| > n^{-c}\right) \lesssim n^{c-1} \log m + n^{c-1/2} \sqrt{\log m}.$$

Note that by the definition of III ,

$$(A.2) \quad |III| \leq \max_{1 \leq p, q \leq m} |\bar{S}_{pq}|^5 \sum_{1 \leq p < q \leq m} \sum_{\boldsymbol{\lambda}: |\boldsymbol{\lambda}|=5} \frac{|\rho_{pq} + k_{pq}\bar{S}_{pq}|^{2-\lambda_1}}{|1 + k_{pq}\bar{S}_{pp}|^{1+\lambda_2} |1 + k_{pq}\bar{S}_{qq}|^{1+\lambda_3}}$$

for $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$. If $\max_{1 \leq p, q \leq m} |\bar{S}_{pq}| \leq n^{-c}$, for all $1 \leq p, q \leq m$ it must be true that

$$(A.3) \quad |\rho_{pq} + k_{pq}\bar{S}_{pq}| \leq 1 + n^{-c}, |1 + k_{pq}\bar{S}_{pp}| \geq 1 - n^{-c}$$

since $k_{pq} \in (0, 1)$. Combining (A.1), (A.2), (A.3), with probability larger than $1 - C(n^{c-1} \log m + n^{c-1/2} \sqrt{\log m})$

$$|III| \leq C n^{-5c} \frac{m(m-1)}{2} \frac{(1 + n^{-c})^2}{(1 - n^{-c})^7} \leq C \frac{m^2}{n^{5c}}$$

for large m, n .

APPENDIX B. TECHNICAL TOOLS

In this section we will lay out the technical tools required to finish the proofs in the paper.

Lemma B.1. *Let f be as defined in (2.2). For any $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{N}_{\geq 0}^3$*

$$\partial^\lambda f(u_1, u_2, u_3) = \begin{cases} (-1)^{\lambda_1 + \lambda_2} \lambda_1! \lambda_2! \frac{u_3^2}{u_1^{1+\lambda_1} u_2^{1+\lambda_2}} & \text{if } \lambda_3 = 0 \\ 2(-1)^{\lambda_1 + \lambda_2} \lambda_1! \lambda_2! \frac{u_3^{2-\lambda_3}}{u_1^{1+\lambda_1} u_2^{1+\lambda_2}} & \text{if } \lambda_3 = 1, 2 \\ 0 & \text{if } \lambda_3 > 2 \end{cases}$$

Theorem B.2 (Isserlis (1918)). *For any natural number $k \geq 1$, let (Z_1, \dots, Z_{2k}) be a mean zero normal vector with covariance matrix $\mathbf{R} = (\rho_{pq})_{1 \leq p, q \leq 2k}$. Then*

$$\mathbb{E}[Z_1 \dots Z_{2k}] = \sum \rho_{p_1 p_2} \dots \rho_{p_{2k-1} p_{2k}},$$

where the summation is over all possible $\frac{(2k)!}{2^k k!}$ partitions of the indices $1, \dots, 2k$ into k pairs $(p_1, p_2), \dots, (p_{2k-1}, p_{2k})$.

Corollary B.3. *For any four indices $1 \leq p_1, q_1, p_2, q_2 \leq m$,*

$$\mathcal{M}_{(p_d, q_d)} := \mathbb{E}[(X_{p_1} X_{q_1} - \rho_{p_1 q_1})(X_{p_2} X_{q_2} - \rho_{p_2 q_2})] = \rho_{p_1 q_2} \rho_{q_1 p_2} + \rho_{p_1 p_2} \rho_{q_1 q_2}$$

Proof. A simple corollary of Theorem B.2. \square

Lemma B.4. *For a fixed natural number k , suppose $1 \leq p_d, q_d \leq m$, $d = 1, \dots, 2k$ are any $2k$ pairs of variable indices. Then*

$$\mathbb{E} \left[\prod_{d=1}^{2k} \bar{S}_{p_d q_d} \right] = O(n^{-k}),$$

where the $O(\cdot)$ term is uniform for all choices of $1 \leq p_d, q_d \leq m$, $d = 1, \dots, 2k$.

Proof. On expansion,

$$\mathbb{E} \left[\prod_{d=1}^{2k} \bar{S}_{p_d q_d} \right] = n^{-2k} \left\{ \sum_{\substack{i_d=1, \dots, n \\ d=1, \dots, 2k}} \mathbb{E} \left[\prod_{d=1}^{2k} (X_{p_d i_d} X_{q_d i_d} - \rho_{p_d q_d}) \right] \right\},$$

so we only need to show the term in $\{\cdot\}$ on the right hand side above is a uniform $O(n^k)$ term. We note that, by independence, an expectation on the right hand of the preceding display can only be non-zero if

$$(B.1) \quad \nexists \quad d' \in [2k] \text{ such that } i_{d'} \neq i_d \quad \forall \quad d \in [2k] \setminus \{d'\}.$$

One way that (B.1) may happen is when there is a permutation $\pi \in \mathcal{S}_{2k}$ such that

$$(B.2) \quad i_{\pi_d} = i_{\pi_k + d}, \text{ for all } d = 1, \dots, k.$$

There can at most be $O(n^k)$ many combinations of i_1, \dots, i_{2k} satisfying (B.2) since when (B.2) is true, the set $\cup_{d=1}^{2k} \{i_d\}$ can at most have k elements leaving us with $O(\binom{n}{k}) = O(n^k)$ many choices for the combination of i_1, \dots, i_{2k} . We note that

when a configuration in (B.1) is such that the set $\cup_{d=1}^{2k} \{i_d\}$ has cardinality exactly equal to k ,

$$\begin{aligned}
 (B.3) \quad \mathbb{E} \left[\prod_{d=1}^{2k} (X_{p_d i_d} X_{q_d i_d} - \rho_{p_d q_d}) \right] \\
 = \prod_{d=1}^k \mathbb{E} [(X_{p_{\pi_d}} X_{q_{\pi_d}} - \rho_{p_{\pi_d} q_{\pi_d}}) (X_{p_{\pi_d+k}} X_{q_{\pi_d+k}} - \rho_{p_{\pi_d+k} q_{\pi_d+k}})] \\
 = \prod_{d=1}^k (\rho_{p_{\pi_d} q_{\pi_d+k}} \rho_{q_{\pi_d} p_{\pi_d+k}} + \rho_{p_{\pi_d} p_{\pi_d+k}} \rho_{q_{\pi_d} q_{\pi_d+k}}),
 \end{aligned}$$

by Corollary B.3. One can also easily see that there are at most $O(n^{k-1})$ many combinations of i_1, \dots, i_{2k} other than ones satisfying (B.2) that can lead to (B.1). Hence by Theorem B.2 and our assumption at the beginning of Section 2, we have

$$\sum_{\substack{i_d=1, \dots, n \\ d=1, \dots, 2k}} \mathbb{E} \left[\prod_{d=1}^{2k} (X_{p_d i_d} X_{q_d i_d} - \rho_{p_d q_d}) \right] = O(n^k),$$

where the $O(\cdot)$ is uniform for all choices of $1 \leq p_d, q_d \leq m$. \square

The next two lemmas on sums of products of the entries in the population correlation matrix $\mathbf{R} = (\rho_{pq})_{1 \leq p, q \leq m}$ are keys for finishing our proofs.

Lemma B.5. *Suppose $\pi = (\pi_1, \dots, \pi_4)$ is a particular permutation of the four indices p, q, r, s , say, $\pi = (p, r, s, q)$. The following estimates are true:*

$$(B.4) \quad \sum_{\substack{1 \leq p, q, r, s \leq m \\ p, q, r, s \text{ all distinct}}} |\rho_{pq} \rho_{rs} \rho_{\pi_1 \pi_2} \rho_{\pi_3 \pi_4}| \lesssim \|\mathbf{R} - \mathbf{I}_m\|_F^4$$

$$(B.5) \quad \sum_{\substack{1 \leq p, q, r \leq m \\ p, q, r \text{ all distinct}}} |\rho_{pq} \rho_{qr} \rho_{pr}| \lesssim \|\mathbf{R} - \mathbf{I}_m\|_F^4 + \|\mathbf{R} - \mathbf{I}_m\|_F^2$$

Proof of Lemma B.5. With a slight abuse of notations, the expression “ $r \neq p, q$ ” means that r is a number that is not equal to p nor q .

By the fact that $2|ab| \leq a^2 + b^2$ for all $a, b \in \mathbb{R}$,

$$\begin{aligned}
 \sum_{\substack{1 \leq p, q, r, s \leq m \\ p, q, r, s \text{ all distinct}}} |\rho_{pq} \rho_{rs} \rho_{\pi_1 \pi_2} \rho_{\pi_3 \pi_4}| &\leq \sum_{\substack{1 \leq p, q, r, s \leq m \\ p, q, r, s \text{ all distinct}}} \{(\rho_{pq} \rho_{rs})^2 + (\rho_{\pi_1 \pi_2} \rho_{\pi_3 \pi_4})^2\} \\
 &\leq 2 \sum_{1 \leq p \neq q \leq m} \rho_{pq}^2 \sum_{1 \leq r \neq s \leq m} \rho_{rs}^2 = 2 \|\mathbf{R} - \mathbf{I}_m\|_F^4,
 \end{aligned}$$

which proves (B.4). Similarly,

$$\begin{aligned}
& \sum_{\substack{1 \leq p, q, r \leq m \\ p, q, r \text{ distinct}}} |\rho_{pq} \rho_{qr} \rho_{pr}| = \sum_{1 \leq p \leq m} \sum_{\substack{1 \leq q \leq m \\ q \neq p}} (|\rho_{pq}| \sum_{\substack{1 \leq r \leq m \\ r \neq p, q}} |\rho_{qr} \rho_{pr}|) \\
& \leq \sum_{\substack{1 \leq p, q \leq m \\ p \neq q}} \rho_{pq}^2 + \sum_{\substack{1 \leq p, q \leq m \\ p \neq q}} \left(\sum_{\substack{1 \leq r \leq m \\ r \neq p, q}} |\rho_{qr} \rho_{pr}| \right)^2 \quad \text{by } 2|ab| \leq a^2 + b^2 \text{ for all } a, b \in \mathbb{R} \\
& = \|\mathbf{R} - \mathbf{I}_m\|_F^2 + \sum_{\substack{1 \leq p, q \leq m \\ p \neq q}} \left(\sum_{\substack{1 \leq r \leq m \\ r \neq p, q}} \sum_{\substack{1 \leq s \leq m \\ s \neq p, q}} |\rho_{qr} \rho_{pr} \rho_{qs} \rho_{ps}| \right) \\
& \leq \|\mathbf{R} - \mathbf{I}_m\|_F^2 + 2\|\mathbf{R} - \mathbf{I}_m\|_F^4,
\end{aligned}$$

where the last inequality comes from a similar proof as the one for (B.4). \square

Lemma B.6. For $k = 5, \dots, 8$,

(i)

$$\sum_{\substack{1 \leq p_d, q_d \leq m \\ d=1, \dots, 4 \\ |\cup_{d=1}^4 \{p_d, q_d\}|=k}} \prod_{d=1}^4 |\rho_{p_d q_d}| = O(m^4) \|\mathbf{R} - \mathbf{I}_m\|_F^{k-4}.$$

(ii) If $\pi = (\pi_1, \dots, \pi_8)$ and $\tau = (\tau_1, \dots, \tau_8)$ are two fixed permutations of the eight indices $p_1, q_1, \dots, p_4, q_4$. For instance, π can be equal to, say, $(p_1, p_4, q_3, q_2, p_2, q_1, q_4, p_3)$. Then

$$\sum_{\substack{1 \leq p_d, q_d \leq m \\ d=1, \dots, 4 \\ |\cup_{d=1}^4 \{p_d, q_d\}|=k}} \prod_{d'=1, 3, 5, 7} |\rho_{\pi_{d'} \pi_{d'+1}} \rho_{\tau_{d'} \tau_{d'+1}}| = O(m^{8-k}) \|\mathbf{R} - \mathbf{I}_m\|_F^{2(k-4)}$$

Proof of Lemma B.6. We first note that for (ii), By the inequality that $2|ab| \leq a^2 + b^2$ for all $a, b \in \mathbb{R}$, we have

$$\sum_{\substack{1 \leq p_d, q_d \leq m \\ d=1, \dots, 4 \\ |\cup_{d=1}^4 \{p_d, q_d\}|=k}} \prod_{d'=1, 3, 5, 7} |\rho_{\pi_{d'} \pi_{d'+1}} \rho_{\tau_{d'} \tau_{d'+1}}| \leq 2^{-1} \sum_{\substack{1 \leq p_d, q_d \leq m \\ d=1, \dots, 4 \\ |\cup_{d=1}^4 \{p_d, q_d\}|=k}} \prod_{d=1}^4 \rho_{p_d q_d}^2,$$

hence to show (ii) it suffices to show

$$\text{(B.6)} \quad \sum_{\substack{1 \leq p_d, q_d \leq m \\ d=1, \dots, 4 \\ |\cup_{d=1}^4 \{p_d, q_d\}|=k}} \prod_{d=1}^4 \rho_{p_d q_d}^2 = O(m^{8-k}) \|\mathbf{R} - \mathbf{I}_m\|_F^{2(k-4)}$$

Given $k \in \{5, \dots, 8\}$, when k of the indices $p_1, q_1, \dots, p_4, q_4$ are distinct, it must be the case that there exist $k-4$ pairs of (p_d, q_d) such that all indices from these $k-4$ pairs are distinct elements from $[m]$. Without lost of generality we can assume these $k-4$ pairs to be $(p_1, q_1), \dots, (p_{k-4}, q_{k-4})$, which contains a total of $2k-8$

distinct indices, and for proving (i) and (B.6) it suffices to show, respectively,

$$(B.7) \quad \sum_{\substack{1 \leq p_1, q_1, \dots, p_4, q_4 \leq m \\ p_1, q_1, \dots, p_{k-4}, q_{k-4} \text{ distinct} \\ |\cup_{d=1}^4 \{p_d, q_d\}| = k}} \prod_{d=1}^4 |\rho_{p_d q_d}| = O(m^4) \|\mathbf{R} - \mathbf{I}_m\|_F^{k-4}$$

and

$$(B.8) \quad \sum_{\substack{1 \leq p_1, q_1, \dots, p_4, q_4 \leq m \\ p_1, q_1, \dots, p_{k-4}, q_{k-4} \text{ distinct} \\ |\cup_{d=1}^4 \{p_d, q_d\}| = k}} \prod_{d=1}^4 \rho_{p_d q_d}^2 = O(m^{8-k}) \|\mathbf{R} - \mathbf{I}_m\|_F^{2(k-4)}.$$

As all the ρ 's are bounded in absolute value by 1, summing over the other $k - (2k - 8) = 8 - k$ indices different from $p_1, q_1, \dots, p_{k-4}, q_{k-4}$ results in a $O(m^{8-k})$ term which gives

$$(B.9) \quad \sum_{\substack{1 \leq p_1, q_1, \dots, p_4, q_4 \leq m \\ p_1, q_1, \dots, p_{k-4}, q_{k-4} \text{ distinct} \\ |\cup_{d=1}^4 \{p_d, q_d\}| = k}} \prod_{d=1}^4 |\rho_{p_d q_d}| = O(m^{8-k}) \sum_{\substack{1 \leq p_d, q_d \leq m \\ d=1, \dots, k-4 \\ p_1, q_1, \dots, p_{k-4}, q_{k-4} \\ \text{are distinct}}} \prod_{d=1}^{k-4} |\rho_{p_d q_d}|.$$

Now since

$$\sum_{1 \leq p \neq q \leq m} |\rho_{pq}| \leq O(m) \|\mathbf{R} - \mathbf{I}_m\|_F$$

by standard norm inequality, evaluating the sum on the right hand side of (B.9) we further obtain

$$\sum_{\substack{1 \leq p_1, q_1, \dots, p_4, q_4 \leq m \\ p_1, q_1, \dots, p_{k-4}, q_{k-4} \text{ distinct} \\ |\cup_{d=1}^4 \{p_d, q_d\}| = k}} \prod_{d=1}^4 |\rho_{p_d q_d}| = O(m^{8-k}) O(m^{k-4}) \|\mathbf{R} - \mathbf{I}_m\|_F^{k-4}$$

which is exactly (B.7). Similarly, summing over the other $8 - k$ indices different from $p_1, q_1, \dots, p_{k-4}, q_{k-4}$ on the left hand side of (B.8) results in a $O(m^{8-k})$ term and hence

$$\sum_{\substack{1 \leq p_1, q_1, \dots, p_4, q_4 \leq m \\ p_1, q_1, \dots, p_{k-4}, q_{k-4} \text{ distinct} \\ |\cup_{d=1}^4 \{p_d, q_d\}| = k}} \prod_{d=1}^4 \rho_{p_d q_d}^2 = O(m^{8-k}) \sum_{\substack{1 \leq p_d, q_d \leq m \\ d=1, \dots, k-4 \\ p_1, q_1, \dots, p_{k-4}, q_{k-4} \\ \text{are distinct}}} \prod_{d=1}^{k-4} \rho_{p_d q_d}^2.$$

Since $\sum_{1 \leq p \neq q \leq m} \rho_{pq}^2 = \|\mathbf{R} - \mathbf{I}_m\|_F^2$, we get (B.8) by continuing from the preceding display. \square

APPENDIX C. PROOFS FOR SECTION 3

C.1. Proof of (3.7)-(3.9). First, we show the estimates in (3.7)-(3.9). Note that by Corollary B.3,

$$\begin{aligned}
 \mathbb{S}(k) &= \sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2 \\ |\cup_{d=1}^2 \{p_d, q_d\}|=k}} (\rho_{p_1 q_2} \rho_{q_1 p_2} + \rho_{p_1 p_2} \rho_{q_1 q_2})^2 \\
 (C.1) \quad &= \sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2 \\ |\cup_{d=1}^2 \{p_d, q_d\}|=k}} (\rho_{p_1 q_2}^2 \rho_{q_1 p_2}^2 + \rho_{p_1 p_2}^2 \rho_{q_1 q_2}^2 + 2\rho_{p_1 q_2} \rho_{p_2 q_1} \rho_{p_1 p_2} \rho_{q_1 q_2})
 \end{aligned}$$

for $k = 2, 3, 4$. Also recall that $\rho_{pp} = 1$ for all $1 \leq p \leq m$. We will analyze the sum in (C.1) for different k .

(3.7): When $k = 2$, with $p_d < q_d$ for $d = 1, 2$, it must be that $p_1 = p_2$ and $q_1 = q_2$, and hence from (C.1)

$$\mathbb{S}(2) = \sum_{1 \leq p < q \leq m} (1 + 2\rho_{pq}^2 + \rho_{pq}^4) = \frac{m(m-1)}{2} + O(\|\mathbf{R} - \mathbf{I}_m\|_F^2).$$

since $\sum_{1 \leq p < q \leq m} \rho_{pq}^2 = 2^{-1} \|\mathbf{R} - \mathbf{I}_m\|_F^2$ and $\rho_{pq}^4 \leq \rho_{pq}^2$.

(3.8): When $k = 3$, one possible configuration of $\cup_{d=1}^2 \{p_d, q_d\}$ as a set with cardinality 3 is that

$$(C.2) \quad p_1 = p_2, \quad p_1 < q_1, \quad p_1 < q_2, \quad \text{and} \quad q_1 \neq q_2.$$

Taking a sum just over the terms in (C.1) whose indices p_1, q_1, p_2, q_2 satisfy the configuration (C.2) we get

$$\begin{aligned}
 &\sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2 \\ \cup_{d=1}^2 \{p_d, q_d\} \text{ as in (C.2)}}} (\rho_{p_1 q_2}^2 \rho_{q_1 p_2}^2 + \rho_{p_1 p_2}^2 \rho_{q_1 q_2}^2 + 2\rho_{p_1 q_2} \rho_{p_2 q_1} \rho_{p_1 p_2} \rho_{q_1 q_2}) \\
 &= \sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2 \\ \cup_{d=1}^2 \{p_d, q_d\} \text{ as in (C.2)}}} (\rho_{p_1 q_2}^2 \rho_{p_1 q_1}^2 + \rho_{q_1 q_2}^2 + 2\rho_{p_1 q_2} \rho_{p_1 q_1} \rho_{q_1 q_2}) \quad \text{by } \rho_{p_1 p_1} = 1 \\
 &\lesssim \sum_{\substack{1 \leq p_1, q_1, q_2 \leq m \\ p_1, q_1, q_2 \text{ distinct}}} (\rho_{p_1 q_1}^2 + |\rho_{p_1 q_2} \rho_{p_1 q_1} \rho_{q_1 q_2}|) \\
 &\lesssim m \|\mathbf{R} - \mathbf{I}_m\|_F^2 + \|\mathbf{R} - \mathbf{I}_m\|_F^4,
 \end{aligned}$$

where the second last inequality is true because we enlarged the set of indices p_1, q_1, q_2 we are summing over and used the fact that

$$\sum_{\substack{1 \leq p_1, q_1, q_2 \leq m \\ p_1, q_1, q_2 \text{ distinct}}} \rho_{p_1 q_2}^2 \rho_{p_1 q_1}^2 \lesssim \sum_{\substack{1 \leq p_1, q_1, q_2 \leq m \\ p_1, q_1, q_2 \text{ distinct}}} \rho_{p_1 q_1}^2$$

since any ρ_{pq}^2 is less than 1, and the last inequality follows from (B.5) and that

$$\sum_{\substack{1 \leq p_1, q_1, q_2 \leq m \\ p_1, q_1, q_2 \text{ distinct}}} \rho_{q_1 q_2}^2 \lesssim \sum_{1 \leq p_1 \leq m} \|\mathbf{R} - \mathbf{I}_m\|_F^2 \lesssim m \|\mathbf{R} - \mathbf{I}_m\|_F^2.$$

The same estimates can be proved for other set configurations of $\cup_{d=1}^2 \{p_d, q_d\}$ similar to the one in (C.2). Since there are only finitely many such configurations, we get the estimate in (3.8).

(3.9): By considering different configurations for the set $\cup_{d=1}^2 \{p_d, q_d\}$ with cardinality 4, from (C.1) we have

$$\begin{aligned} \mathbb{S}(4) &\lesssim \sum_{\substack{1 \leq p_1, q_1, p_2, q_2 \leq m \\ p_1, q_1, p_2, q_2 \text{ distinct}}} \rho_{p_1 q_1}^2 \rho_{p_2 q_2}^2 + |\rho_{p_1 q_2} \rho_{q_1 p_2} \rho_{p_1 p_2} \rho_{q_1 q_2}| \\ &\lesssim \|\mathbf{R} - \mathbf{I}_m\|_F^4, \end{aligned}$$

where the last inequality used (B.4) and

$$\sum_{\substack{1 \leq p_1, q_1, p_2, q_2 \leq m \\ p_1, q_1, p_2, q_2 \text{ distinct}}} \rho_{p_1 q_1}^2 \rho_{p_2 q_2}^2 \lesssim \|\mathbf{R} - \mathbf{I}_m\|_F^2 \sum_{\substack{1 \leq p_1, q_1 \leq m \\ p_1, q_1 \text{ distinct}}} \rho_{p_1 q_1}^2 \lesssim \|\mathbf{R} - \mathbf{I}_m\|_F^4.$$

C.2. Proof of (3.17). In fact, the strategy we used in proving (3.8) will also lead to a quick proof of the estimates for $\mathbb{T}(k)$, $k = 5, \dots, 8$ in (3.17). By definition,

$$\mathbb{T}(k) = \sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2,3,4 \\ |\cup_{d=1}^4 \{p_d, q_d\}|=k}} \mathbb{E} \left[\prod_{d \in [4]} (X_{p_d} X_{q_d} - \rho_{p_d q_d}) \right].$$

By expanding the product $\prod_{d \in [4]} (X_{p_d} X_{q_d} - \rho_{p_d q_d})$ at the end of the above equation and taking expectation with respect to Theorem B.2, one can see that

$$(C.3) \quad \mathbb{T}(k) \lesssim \sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2,3,4 \\ |\cup_{d=1}^4 \{p_d, q_d\}|=k}} \sum_{\pi \in \mathcal{S}_8} |\rho_{\pi_1 \pi_2} \rho_{\pi_3 \pi_4} \rho_{\pi_5 \pi_6} \rho_{\pi_7 \pi_8}|,$$

where here we interpret $\pi = (\pi_1, \dots, \pi_8)$ as a permutation of the eight indices $p_1, q_1, \dots, p_4, q_4$. When the permutation $\pi = (p_1, q_1, p_2, q_2, p_3, q_3, p_4, q_4)$, we have

$$\begin{aligned} (C.4) \quad &\sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2,3,4 \\ |\cup_{d=1}^4 \{p_d, q_d\}|=k}} |\rho_{\pi_1 \pi_2} \rho_{\pi_3 \pi_4} \rho_{\pi_5 \pi_6} \rho_{\pi_7 \pi_8}| \\ &\leq \sum_{\substack{1 \leq p_d, q_d \leq m \\ d=1,2,3,4 \\ \text{and} \\ k \text{ of } p_1, q_1, \dots, p_4, q_4 \\ \text{are distinct}}} \prod_{d=1}^4 |\rho_{p_d q_d}| = O(m^4) \|\mathbf{R} - \mathbf{I}_m\|_F^{k-4} \end{aligned}$$

by Lemma B.6(i). Although (C.4) is only proved for $\pi = (p_1, q_1, p_2, \dots, p_4, q_4)$, a same bound for all other permutations easily generalize, which gives our estimate in (3.17) in light of (C.3).

C.3. Proof of (3.27)-(3.29).

(3.27): We first write

$$(C.5) \quad \sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2,3,4}} \mathbb{P}_1 = \sum_{k=5}^8 \left\{ \sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2,3,4: \\ |\cup_{d=1}^4 \{p_d, q_d\}|=k}} \mathbb{P}_1 \right\} + O(m^4)$$

where the $O(m^4)$ term comes from a remaining sum of $O(m^4)$ many universally bounded terms when $|\cup_{d=1}^4 \{p_d, q_d\}| \leq 4$. By the definition of \mathbb{P}_1 in (3.20), Corollary B.3 and Lemma B.6(i), it can be seen that for each $k = 5, \dots, 8$,

$$\sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2,3,4: \\ |\cup_{d=1}^4 \{p_d, q_d\}|=k}} \mathbb{P}_1 \lesssim \sum_{\substack{1 \leq p_d, q_d \leq m \\ d=1,2,3,4 \\ |\cup_{d=1}^4 \{p_d, q_d\}|=k}} \prod_{d=1}^4 |\rho_{p_d q_d}| = O(m^4) \|\mathbf{R} - \mathbf{I}_m\|_F^{k-4},$$

giving (3.27) in light of (C.5).

(3.28) and (3.29): Similar to (C.5) for $u = 1, 2, 3$, we can write

$$(C.6) \quad \sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2,3,4}} \mathbb{P}_1 \mathbb{P}_u = \sum_{k=4}^8 \sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2,3,4 \\ |\cup_{d=1}^4 \{p_d, q_d\}|=k}} \mathbb{P}_1 \mathbb{P}_u + O(m^3).$$

By Corollary B.3 we get that $\mathbb{P}_1 \mathbb{P}_u$ is a finite sum of terms each having the form

$$\prod_{d'=1,3,5,7} \rho_{\pi_{d'} \pi_{d'+1}} \prod_{d'=1,3,5,7} \rho_{\tau_{d'} \tau_{d'+1}}$$

for $\pi = (\pi_1, \dots, \pi_8)$ and $\tau = (\tau_1, \dots, \tau_8)$ that are certain permutations of the 8 indices $p_1, q_1, \dots, p_4, q_4$. As such, by Lemma B.6(ii), for a given $k = 5, \dots, 8$,

$$(C.7) \quad \sum_{k=5}^8 \sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2,3,4 \\ |\cup_{d=1}^4 \{p_d, q_d\}|=k}} \mathbb{P}_1 \mathbb{P}_u \lesssim \sum_{k=5}^8 O(m^{8-k}) \|\mathbf{R} - \mathbf{I}_m\|_F^{2(k-4)} \\ = \sum_{k=0}^3 O(m^k) \|\mathbf{R} - \mathbf{I}_m\|_F^{2(4-k)}.$$

Given (C.6) and (C.7) it remains to show

$$(C.8) \quad \sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2,3,4 \\ |\cup_{d=1}^4 \{p_d, q_d\}|=4}} \mathbb{P}_1^2 = \binom{m}{2} \binom{m-2}{2} + O(m^3) \|\mathbf{R} - \mathbf{I}_m\|_F$$

and, for $u = 2, 3$,

$$(C.9) \quad \sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2,3,4 \\ |\cup_{d=1}^4 \{p_d, q_d\}|=4}} \mathbb{P}_1 \mathbb{P}_u = O(m^3) \|\mathbf{R} - \mathbf{I}_m\|_F$$

to prove (3.28) and (3.29). To that end we make the following claim:

Claim. Suppose $\pi = (\pi_1, \dots, \pi_8)$ and $\tau = (\tau_1, \dots, \tau_8)$ are two given permutations of eight indices $p_1, q_1, \dots, p_4, q_4 \in [m]$. Then

$$(C.10) \quad \sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1, \dots, 4 \\ |\cup_{d=1}^4 \{p_d, q_d\}|=4}} \left(\prod_{d'=1,3,5,7} \rho_{\pi_{d'} \pi_{d'+1}} \rho_{\tau_{d'} \tau_{d'+1}} \right) = O(m^3) \|\mathbf{R} - \mathbf{I}_m\|_F$$

unless, as elements in $[m]$,

$$(C.11) \quad \pi_{d'} = \pi_{d'+1}, \quad \tau_{d'} = \tau_{d'+1}$$

for all $d' = 1, 3, 5, 7$ when $1 \leq p_d < q_d \leq m$ for all $d = 1, \dots, 4$ and $|\cup_{d=1}^4 \{p_d, q_d\}| = 4$.

The proof of this claim will be left till the end of this section. Using this, we will first show (C.9) for $u = 2$ while the proof for $u = 3$ follows similarly and is thus omitted.

By Corollary B.3, on expansion we get that the $\sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2,3,4 \\ |\cup_{d=1}^4 \{p_d, q_d\}|=4}} \mathbb{P}_1 \mathbb{P}_2$ is a finite sum of terms each having the form as in the left hand side of (C.10) with π and τ NOT satisfying the description in (C.11) of the claim. For example, by Corollary B.3, on expansion

$$\begin{aligned} \mathbb{P}_1 &= (\rho_{p_1 q_2} \rho_{q_1 p_2} + \rho_{p_1 p_2} \rho_{q_1 q_2})(\rho_{p_3 q_4} \rho_{q_3 p_4} + \rho_{p_3 p_4} \rho_{q_3 q_4}) = \rho_{p_1 p_2} \rho_{q_1 q_2} \rho_{p_3 p_4} \rho_{q_3 q_4} + \dots \\ \mathbb{P}_2 &= (\rho_{p_1 q_3} \rho_{q_1 p_3} + \rho_{p_1 p_3} \rho_{q_1 q_3})(\rho_{p_2 q_4} \rho_{q_2 p_4} + \rho_{p_2 p_4} \rho_{q_2 q_4}) = \rho_{p_1 p_3} \rho_{q_1 q_3} \rho_{p_2 p_4} \rho_{q_2 q_4} + \dots, \end{aligned}$$

which leads to

$$(C.12) \quad \sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2,3,4 \\ |\cup_{d=1}^4 \{p_d, q_d\}|=4}} \mathbb{P}_1 \mathbb{P}_2 = \sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2,3,4 \\ |\cup_{d=1}^4 \{p_d, q_d\}|=4}} \left(\prod_{d=1,3,5,7} \rho_{\pi'_d \pi'_{d+1}} \rho_{\tau'_d \tau'_{d+1}} \right) + \dots,$$

where

$$(C.13) \quad \pi' = (\pi'_1, \dots, \pi'_8) := (p_1, p_2, q_1, q_2, p_3, p_4, q_3, q_4)$$

and

$$(C.14) \quad \tau' = (\tau'_1, \dots, \tau'_8) := (p_1, p_3, q_1, q_3, p_2, p_4, q_2, q_4)$$

and similar terms are omitted in \dots above. Note that when $|\cup_{d=1}^4 \{p_d, q_d\}| = 4$ and $1 \leq p_d < q_d \leq m$, there must be a pair among $\{(\pi'_d, \pi'_{d+1}), (\tau'_d, \tau'_{d+1}) : d = 1, 3, 5, 7\}$ that contains two distinct elements in $[m]$ due to a mismatch of the permutations π' and τ' : For if not in consideration of π' it must be the case that $p_1 = p_2$, $q_1 = q_2$, $p_3 = p_4$ and $q_3 = q_4$ with p_1, q_1, p_3, p_4 being four distinct elements in $[m]$, but this will imply $\tau'_1 = p_1 \neq p_3 = \tau'_2$, a contradiction. By the claim above the first term on the right hand side of (C.12) equals to $O(m^3) \|\mathbf{R} - \mathbf{I}_m\|_F$, where as the

finitely many omitted terms \dots in (C.12) can also be similarly bounded and (C.9) is proved.

We now show (C.8), again with Corollary B.3, we expand

$$(C.15) \quad \sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2,3,4 \\ |\cup_{d=1}^4 \{p_d, q_d\}|=4}} \mathbb{P}_1^2 = \sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2,3,4 \\ |\cup_{d=1}^4 \{p_d, q_d\}|=4}} \left\{ (\rho_{p_1 p_2} \rho_{q_1 q_2} \rho_{p_3 p_4} \rho_{q_3 q_4})^2 + \right. \\ \left. (\rho_{p_1 p_2} \rho_{q_1 q_2} \rho_{p_3 q_4} \rho_{q_3 p_4})^2 + (\rho_{p_1 q_2} \rho_{q_1 p_2} \rho_{p_3 p_4} \rho_{q_3 q_4})^2 + (\rho_{p_1 q_2} \rho_{q_1 p_2} \rho_{p_3 q_4} \rho_{q_3 p_4})^2 \right\} + \dots,$$

where we leave it to the reader to check that the omitted terms in \dots of (C.15) is of order $O(m^3) \|\mathbf{R} - \mathbf{I}_m\|_F$ due to mismatch of permutations as in (C.13) and (C.14). In fact, summing over the three terms on the second line of (C.15) also contribute a term of order $O(m^3) \|\mathbf{R} - \mathbf{I}_m\|_F$: For example, summing over the last term on the second line of (C.15) equals

$$(C.16) \quad \sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2,3,4 \\ |\cup_{d=1}^4 \{p_d, q_d\}|=4}} \left(\prod_{d'=1,3,5,7} \rho_{\tilde{\pi}_{d'} \tilde{\pi}_{d'+1}}^2 \right)$$

with

$$\tilde{\pi} = (\tilde{\pi}_1, \dots, \tilde{\pi}_8) = (p_1, q_2, q_1, p_2, p_3, q_4, q_3, p_4).$$

When $1 \leq p_d < q_d \leq m$ and $|\cup_{d=1}^4 \{p_d, q_d\}| = 4$, we cannot have $\tilde{\pi}_{d'} = \tilde{\pi}_{d'+1}$ for all $d' = 1, 3, 5, 7$ and hence by the previous claim (C.16) is of order $O(m^3) \|\mathbf{R} - \mathbf{I}_m\|_F$. Hence it remains to show that summing over the terms on the first line of (C.15) gives

$$(C.17) \quad \sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2,3,4 \\ |\cup_{d=1}^4 \{p_d, q_d\}|=4}} (\rho_{p_1 p_2} \rho_{q_1 q_2} \rho_{p_3 p_4} \rho_{q_3 q_4})^2 = \binom{m}{2} \binom{m-2}{2} + O(m^3) \|\mathbf{R} - \mathbf{I}_m\|_F$$

When $|\cup_{d=1}^4 \{p_d, q_d\}| = 4$ with $p_d < q_d$ for all $d = 1, \dots, 4$, as a set $\cup_{d=1}^4 \{p_d, q_d\}$ can take the configuration

$$(C.18) \quad p_1 = p_2, \quad q_1 = q_2, \quad p_3 = p_4, \quad q_3 = q_4.$$

When (C.18) is true, $\rho_{p_1 p_2}^2 \rho_{q_1 q_2}^2 \rho_{p_3 p_4}^2 \rho_{q_3 q_4}^2 = 1$, and hence

$$\begin{aligned}
& \sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2,3,4 \\ \cup_{d=1}^4 \{p_d, q_d\} = 4}} \rho_{p_1 p_2}^2 \rho_{q_1 q_2}^2 \rho_{p_3 p_4}^2 \rho_{q_3 q_4}^2 \\
&= \sum_{\substack{1 \leq p_3 < q_3 \leq m \\ \{p_1, q_1\} \cap \{p_3, q_3\} = \emptyset}} \sum_{1 \leq p_1 < q_1 \leq m} 1 + \sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2,3,4 \\ |\cup_{d=1}^4 \{p_d, q_d\}| = 4 \\ \cup_{d=1}^4 \{p_d, q_d\} \text{ NOT as in (C.18)}}} \rho_{p_1 p_2}^2 \rho_{q_1 q_2}^2 \rho_{p_3 p_4}^2 \rho_{q_3 q_4}^2 \\
&\text{(C.19)} \\
&= \binom{m}{2} \binom{m-2}{2} + \sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2,3,4 \\ |\cup_{d=1}^4 \{p_d, q_d\}| = 4 \\ \cup_{d=1}^4 \{p_d, q_d\} \text{ NOT as in (C.18)}}} \rho_{p_1 p_2}^2 \rho_{q_1 q_2}^2 \rho_{p_3 p_4}^2 \rho_{q_3 q_4}^2.
\end{aligned}$$

For any configurations of the set $\cup_{d=1}^4 \{p_d, q_d\}$ other than (C.18), one of (i) $p_1 \neq p_2$, (ii) $q_1 \neq q_2$, (iii) $p_3 \neq p_4$ or (iv) $q_3 \neq q_4$ must be true. For example, one such configuration is

$$(C.20) \quad p_1 < p_2 = q_1 < q_2 = p_3 < p_4 = q_3 < q_4.$$

For this particular configuration, (i) $p_1 \neq p_2$ is true. Then we leave it to the reader to verify that by the same line of reasoning as in the proof of the claim below, we can show

$$\sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2,3,4 \\ |\cup_{d=1}^4 \{p_d, q_d\}| = 4 \\ \cup_{d=1}^4 \{p_d, q_d\} \text{ as in (C.20)}}} \rho_{p_1 p_2}^2 \rho_{q_1 q_2}^2 \rho_{p_3 p_4}^2 \rho_{q_3 q_4}^2 = O(m^3) \|\mathbf{R} - \mathbf{I}_m\|_F,$$

where similar bounds can in fact be proved for all configurations of $\cup_{d=1}^4 \{p_d, q_d\}$ other than (C.18). This, together with (C.19), leads to (C.17).

Proof of the claim. Suppose (C.11) is not true for some $d' \in \{1, 3, 5, 7\}$, and without loss of generality we assume $\pi_1 \neq \pi_2$. Since $|\rho_{pq}| \leq 1$ for all $1 \leq p, q \leq m$, we have

$$\begin{aligned}
& \sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1, \dots, 4 \\ |\cup_{d=1}^4 \{p_d, q_d\}| = 4}} \prod_{d'=1,3,5,7} |\rho_{\pi_{d'} \pi_{d'+1}} \rho_{\tau_{d'} \tau_{d'+1}}| \leq \sum_{\substack{1 \leq \pi_d, \tau_d \leq m \\ d=1, \dots, 8 \\ \pi_1 \neq \pi_2 \\ |\cup_{d=1}^8 (\{\pi_d\} \cup \{\tau_d\})| = 4}} \prod_{d'=1,3,5,7} |\rho_{\pi_{d'} \pi_{d'+1}} \rho_{\tau_{d'} \tau_{d'+1}}| \\
&\leq \sum_{\substack{1 \leq \pi_d, \tau_d \leq m, d=1, \dots, 8 \\ \pi_1 \neq \pi_2 \\ |\cup_{d=1}^8 (\{\pi_d\} \cup \{\tau_d\})| = 4}} |\rho_{\pi_1 \pi_2}| = O(m^2) \sum_{1 \leq \pi_1 \neq \pi_2 \leq m} |\rho_{\pi_1 \pi_2}| = O(m^3) \|\mathbf{R} - \mathbf{I}_m\|_F,
\end{aligned}$$

aa desired. □

APPENDIX D. PROOFS FOR SECTION 4

Before finishing the proofs for Section 4, we first give the definition of the kernel $h_{3,pq}$ as mentioned in the main text:

$$\begin{aligned}
& h_{3,pq}(\mathbf{X}_{pq,i}, \mathbf{X}_{pq,j}, \mathbf{X}_{pq,k}, \mathbf{X}_{pq,l}) \\
& := \underbrace{n^{-4} \binom{n}{4}}_{O(1)} \sum_{\pi \in \mathcal{S}_4} \left\{ (X_{p\pi(i)}^2 - 1)(X_{p\pi(j)}^2 - 1)(X_{p\pi(k)}X_{q\pi(k)} - \rho_{pq})(X_{p\pi(l)}X_{q\pi(l)} - \rho_{pq}) \right\} \\
& + \underbrace{\frac{n^{-4}}{\binom{n-3}{1}} \binom{n}{4}}_{O(n^{-1})} \sum_{\substack{\{i',j',k',l'\} \\ \subset \{i,j,k,l\} \\ i',j',k' \text{ distinct}}} \sum_{\pi \in \mathcal{S}_3} \left\{ (X_{p\pi(i')}X_{q\pi(i')} - \rho_{pq})^2 (X_{p\pi(j')}^2 - 1)(X_{p\pi(k')}^2 - 1) \right. \\
& \quad + (X_{p\pi(i')}^2 - 1)^2 (X_{p\pi(j')}X_{q\pi(j')} - \rho_{pq})(X_{p\pi(k')}X_{q\pi(k')} - \rho_{pq}) \\
& \quad \left. + 4(X_{p\pi(i')}^2 - 1)(X_{p\pi(i')}X_{q\pi(i')} - \rho_{pq})(X_{p\pi(j')}^2 - 1)(X_{p\pi(k')}X_{q\pi(k')} - \rho_{pq}) \right\} \\
& + \underbrace{\frac{n^{-4}}{\binom{n-2}{2}} \binom{n}{4}}_{O(n^{-2})} \sum_{\substack{\{i',j'\} \\ \subset \{i,j,k,l\} \\ i',j' \text{ distinct}}} \sum_{\pi \in \mathcal{S}_2} \left\{ (X_{p\pi(i')}^2 - 1)^2 (X_{p\pi(j')}X_{q\pi(j')} - \rho_{pq})^2 \right. \\
& \quad + 2(X_{p\pi(i')}^2 - 1)(X_{p\pi(j')}^2 - 1)(X_{p\pi(j')}X_{q\pi(j')} - \rho_{pq})^2 \\
& \quad + 2(X_{p\pi(i')}X_{q\pi(i')} - \rho_{pq})(X_{p\pi(j')}X_{q\pi(j')} - \rho_{pq})(X_{p\pi(j')}^2 - 1)^2 \\
& \quad \left. + 2(X_{p\pi(i')}^2 - 1)(X_{p\pi(i')}X_{q\pi(i')} - \rho_{pq})(X_{p\pi(j')}^2 - 1)(X_{p\pi(j')}X_{q\pi(j')} - \rho_{pq}) \right\} \\
& + \underbrace{\frac{n^{-4}}{\binom{n-1}{3}} \binom{n}{4}}_{O(n^{-3})} \sum_{i' \in \{i,j,k,l\}} \{ (X_{pi'}^2 - 1)^2 (X_{pi'}X_{qi'} - \rho_{pq})^2 \}
\end{aligned}$$

We now proceed with the remaining proofs.

Proof for Lemma 4.2. Note that by definition,

$$\text{(D.1)} \quad II_2 = \sum_{1 \leq p < q \leq m} \bar{S}_{pp} \bar{S}_{qq} \bar{S}_{pq}^2 + \sum_{\substack{\boldsymbol{\lambda} \in \mathbb{N}_{\geq 0}^3: \\ 1 \leq |\boldsymbol{\lambda}| \leq 4 \\ \lambda_3 \neq 2}} \left\{ \sum_{1 \leq p < q \leq m} \frac{\partial^{\boldsymbol{\lambda}} f(1, 1, \rho_{pq})}{\boldsymbol{\lambda}!} \bar{S}_{pp}^{\lambda_1} \bar{S}_{qq}^{\lambda_2} \bar{S}_{pq}^{\lambda_3} \right\}.$$

Since there are only finitely many $\boldsymbol{\lambda}$ we are summing over for the second term in (D.1), by the general fact that $2|ab| \leq a^2 + b^2$, it suffices to show that, for $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ with $1 \leq |\boldsymbol{\lambda}| \leq 4$ and $\lambda_3 \neq 2$, the quantities

$$\text{(D.2)} \quad \mathbb{E} \left[\left(\sum_{1 \leq p < q \leq m} \frac{\partial^{\boldsymbol{\lambda}} f(1, 1, \rho_{pq})}{\boldsymbol{\lambda}!} \bar{S}_{pp}^{\lambda_1} \bar{S}_{qq}^{\lambda_2} \bar{S}_{pq}^{\lambda_3} \right)^2 \right],$$

as well as

$$(D.3) \quad \mathbb{E} \left[\left(\sum_{1 \leq p < q \leq m} \bar{S}_{pp} \bar{S}_{qq} \bar{S}_{pq}^2 \right)^2 \right],$$

can be bounded by the right hand side of (4.3) up to some multiplicative constants. We will first show it for (D.2) case by case according to the multi-index degree of λ . The arguments rely on the fact that, by Lemma B.1, it must be true that

$$(D.4) \quad |\partial^\lambda f(1, 1, \rho_{pq})| \leq C |\rho_{pq}| \quad \text{when} \quad \lambda_3 = 1$$

and

$$(D.5) \quad |\partial^\lambda f(1, 1, \rho_{pq})| \leq C \rho_{pq}^2 \quad \text{when} \quad \lambda_3 = 0$$

for some constant $C > 0$. Consider 3 cases:

$|\lambda| = 3$ or 4: With the facts in (D.4) and (D.5), with Lemma B.4, (D.2) is less than

$$(D.6) \quad O(n^{-|\lambda|}) \sum_{\substack{1 \leq p < q \leq m \\ 1 \leq r < s \leq m}} |\rho_{pq}| |\rho_{rs}| \quad \text{or} \quad O(n^{-|\lambda|}) \sum_{\substack{1 \leq p < q \leq m \\ 1 \leq r < s \leq m}} |\rho_{pq}|^2 |\rho_{rs}|^2.$$

Respectively, by properties of norms they can be estimated by

$$O(n^{-|\lambda|} m^2) \|\mathbf{R} - \mathbf{I}_m\|_F^2 \quad \text{and} \quad O(n^{-|\lambda|}) \|\mathbf{R} - \mathbf{I}_m\|_F^4,$$

which are both less than the right hand side of (4.3) up to constants since $|\lambda| = 3$ or 4.

$|\lambda| = 1$: The only $\lambda \in \mathbb{N}_{\geq 0}^3$ with $|\lambda| = 1$ and $\lambda_3 \neq 2$ are $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$. When $\lambda_3 = 0$, by (D.5) and Lemma B.4 the second moment quantity in (D.2) is bounded by

$$O(n^{-1}) \sum_{\substack{1 \leq p < q \leq m \\ 1 \leq r < s \leq m}} \rho_{pq}^2 \rho_{rs}^2 = O(n^{-1}) \|\mathbf{R} - \mathbf{I}_m\|_F^4,$$

less than the right hand side of (4.3). When $\lambda = (0, 0, 1)$, by Lemma B.1, (D.2) equals

$$(D.7) \quad \begin{aligned} \mathbb{E} \left[\left(\sum_{1 \leq p < q \leq m} 2\rho_{pq} \bar{S}_{pq} \right)^2 \right] &= 4 \sum_{\substack{1 \leq p < q \leq m \\ 1 \leq r < s \leq m}} \rho_{pq} \rho_{rs} \mathbb{E}[\bar{S}_{pq} \bar{S}_{rs}] \\ &= 4n^{-1} \left(\sum_{\substack{1 \leq p < q \leq m \\ 1 \leq r < s \leq m}} \rho_{pq} \rho_{rs} \rho_{ps} \rho_{qr} + \sum_{\substack{1 \leq p < q \leq m \\ 1 \leq r < s \leq m}} \rho_{pq} \rho_{rs} \rho_{pr} \rho_{qs} \right) \\ &= O(n^{-1}) (\|\mathbf{R} - \mathbf{I}_m\|_F^4 + \|\mathbf{R} - \mathbf{I}_m\|_F^2), \end{aligned}$$

where the second equality comes from the fact that

$$\mathbb{E}[\bar{S}_{pq} \bar{S}_{rs}] = n^{-1} \mathbb{E}[(X_p X_q - \rho_{pq})(X_r X_s - \rho_{rs})] = n^{-1} (\rho_{ps} \rho_{qr} + \rho_{pr} \rho_{qs})$$

due to the i.i.d.'ness of samples and Corollary B.3. To show the last equality, by exploiting symmetry it is easy to see that

$$(D.8) \quad \sum_{\substack{1 \leq p < q \leq m \\ 1 \leq r < s \leq m}} \rho_{pq} \rho_{rs} \rho_{ps} \rho_{qr} + \sum_{\substack{1 \leq p < q \leq m \\ 1 \leq r < s \leq m}} \rho_{pq} \rho_{rs} \rho_{pr} \rho_{qs} \\ \lesssim \sum_{k=2}^4 \sum_{\substack{1 \leq p, q, r, s \leq m \\ |\{p\} \cup \{q\} \cup \{r\} \cup \{s\}| = k}} |\rho_{pq} \rho_{rs} \rho_{pr} \rho_{qs}|.$$

Observe that

$$\begin{aligned} \sum_{\substack{1 \leq p, q, r, s \leq m \\ |\{p\} \cup \{q\} \cup \{r\} \cup \{s\}| = 2}} |\rho_{pq} \rho_{rs} \rho_{pr} \rho_{qs}| &\lesssim \sum_{1 \leq p < q \leq m} \rho_{pq}^2 \\ \sum_{\substack{1 \leq p, q, r, s \leq m \\ |\{p\} \cup \{q\} \cup \{r\} \cup \{s\}| = 3}} |\rho_{pq} \rho_{rs} \rho_{pr} \rho_{qs}| &\lesssim \sum_{\substack{1 \leq p, q, r \leq m \\ p, q, r \text{ distinct}}} |\rho_{pq} \rho_{pr} \rho_{qr}| \\ \sum_{\substack{1 \leq p, q, r, s \leq m \\ |\{p\} \cup \{q\} \cup \{r\} \cup \{s\}| = 4}} |\rho_{pq} \rho_{rs} \rho_{pr} \rho_{qs}| &= \sum_{\substack{1 \leq p, q, r, s \leq m \\ p, q, r, s \text{ distinct}}} |\rho_{pq} \rho_{rs} \rho_{pr} \rho_{qs}|. \end{aligned}$$

In light of Lemma B.5, applying these bounds to (D.8) implies (D.7).

$|\lambda| = 2$: The only λ 's with $|\lambda| = 2$ and $\lambda_3 \neq 2$ are $(1, 1, 0)$, $(2, 0, 0)$, $(0, 2, 0)$, $(1, 0, 1)$ and $(0, 1, 1)$. For the first three of these since $\lambda_3 = 0$, by (D.5) and Lemma B.4 the quantity in (D.2) equals

$$(D.9) \quad O(n^{-2}) \sum_{\substack{1 \leq p < q \leq m \\ 1 \leq r < s \leq m}} \rho_{pq}^2 \rho_{rs}^2 = O(n^{-2}) \|\mathbf{R} - \mathbf{I}_m\|_F^4.$$

For $\lambda = (1, 0, 1)$, with Lemma B.1 the quantity in (D.2) equals

$$(D.10) \quad 4 \sum_{\substack{1 \leq p < q \leq m \\ 1 \leq r < s \leq m}} \rho_{pq} \rho_{rs} \mathbb{E}[\bar{S}_{pp} \bar{S}_{pq} \bar{S}_{rr} \bar{S}_{rs}].$$

By simple argument as in the proof of Lemma B.4 and Corollary B.3, it is not hard to see that

$$\begin{aligned} &\mathbb{E}[\bar{S}_{pp} \bar{S}_{pq} \bar{S}_{rr} \bar{S}_{rs}] \\ &= \frac{n(n-1)}{n^4} \left\{ \mathbb{E}[(X_p^2 - 1)(X_r^2 - 1)] \mathbb{E}[(X_p X_q - \rho_{pq})(X_r X_s - \rho_{rs})] + \right. \\ &\quad \mathbb{E}[(X_p^2 - 1)(X_p X_q - \rho_{pq})] \mathbb{E}[(X_r^2 - 1)(X_r X_s - \rho_{rs})] + \\ &\quad \left. \mathbb{E}[(X_p^2 - 1)(X_r X_s - \rho_{rs})] \mathbb{E}[(X_r^2 - 1)(X_p X_q - \rho_{pq})] \right\} + O(n^{-3}) \\ (D.11) \quad &= O(n^{-2})(2\rho_{pr}^2(\rho_{ps}\rho_{qr} + \rho_{pr}\rho_{qs}) + 4\rho_{pq}\rho_{rs} + 4\rho_{pr}\rho_{ps}\rho_{rp}\rho_{rq}) + O(n^{-3}). \end{aligned}$$

Substituting (D.11) into (D.10) we get

$$\begin{aligned}
& 4 \sum_{\substack{1 \leq p < q \leq m \\ 1 \leq r < s \leq m}} \rho_{pq} \rho_{rs} \mathbb{E}[\bar{S}_{pp} \bar{S}_{pq} \bar{S}_{rr} \bar{S}_{rs}] \\
&= 4 \sum_{\substack{1 \leq p < q \leq m \\ 1 \leq r < s \leq m}} \rho_{pq} \rho_{rs} \{O(n^{-2})(2\rho_{pr}^2(\rho_{ps}\rho_{qr} + \rho_{pr}\rho_{qs}) + 4\rho_{pq}\rho_{rs} + 4\rho_{pr}\rho_{ps}\rho_{rp}\rho_{rq}) + O(n^{-3})\} \\
&\leq O(n^{-2}) \sum_{k=2}^4 \sum_{\substack{1 \leq p, q, r, s \leq m \\ |\{p\} \cup \{q\} \cup \{r\} \cup \{s\}| = k}} |\rho_{pq}\rho_{rs}\rho_{pr}\rho_{qs}| + O(n^{-3}) \left(\sum_{\substack{1 \leq p < q \leq m \\ 1 \leq r < s \leq m}} |\rho_{pq}\rho_{rs}| \right)
\end{aligned}$$

(D.12)

$$\leq O(n^{-2})(\|\mathbf{R} - \mathbf{I}_m\|_F^4 + \|\mathbf{R} - \mathbf{I}_m\|_F^2) + O(m^2 n^{-3}) \|\mathbf{R} - \mathbf{I}_m\|_F^2,$$

where the last two inequalities make use of similar arguments that prove (D.7). By a symmetry argument the same estimate holds for $\lambda = (0, 1, 1)$. Both (D.9) and (D.12) are less than the right hand side of (4.3).

It remains to form an estimate for (D.3). Note that

$$\mathbb{E} \left[\left(\sum_{1 \leq p < q \leq m} \bar{S}_{pp} \bar{S}_{qq} \bar{S}_{pq}^2 \right)^2 \right] = \sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2 \\ |\cup_{d=1}^2 \{p_d, q_d\}|=4}} \mathbb{E} \left[\prod_{d=1}^2 \bar{S}_{p_d p_d} \bar{S}_{q_d q_d} \bar{S}_{p_d q_d}^2 \right] + O\left(\frac{m^3}{n^4}\right),$$

where the $O(n^{-4})$ term comes from an argument similar to the proof of Lemma B.4, and the $O(m^3)$ term comes from that the $O(m^3)$ many choices for p_1, q_1, p_2, q_2 when $|\cup_{d=1}^2 \{p_d, q_d\}| \leq 3$. Hence it now suffices to show the first term on the right hand side of the preceding display is less than the RHS of (4.3). The argument is simple but a little tedious so we just sketch it here: By a similar argument as in the proof of Lemma B.4 we must have

$$\begin{aligned}
& \sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2 \\ |\cup_{d=1}^2 \{p_d, q_d\}|=4}} \mathbb{E} \left[\prod_{d=1}^2 \bar{S}_{p_d p_d} \bar{S}_{q_d q_d} \bar{S}_{p_d q_d}^2 \right] = O\left(\frac{m^4}{n^5}\right) + \\
& \sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2 \\ |\cup_{d=1}^2 \{p_d, q_d\}|=4}} O\left(\frac{\sum_{k \in \{p_2, q_1, q_2\}} \mathbb{E}[(X_{p_1}^2 - 1)(X_k^2 - 1)] + \sum_{d'=1}^2 \mathbb{E}[(X_{p_1}^2 - 1)(X_{p_{d'}} X_{q_{d'}} - \rho_{p_{d'} q_{d'}})]}{n^4}\right),
\end{aligned}$$

where the expectations on the right come from the fact that $(X_{p_1}^2 - 1)$ must pair with one of $(X_{p_2}^2 - 1)$, $(X_{q_1}^2 - 1)$, $(X_{q_2}^2 - 1)$, $(X_{p_1} X_{q_1} - \rho_{p_1 q_1})$ and $(X_{p_2} X_{q_2} - \rho_{p_2 q_2})$ as in (B.3). By Corollary B.3, for k equals p_2, q_1 or q_2 , it must be that

(D.14)

$$\sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2 \\ |\cup_{d=1}^2 \{p_d, q_d\}|=4}} \mathbb{E}[(X_{p_1}^2 - 1)(X_k^2 - 1)] = \sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2 \\ |\cup_{d=1}^2 \{p_d, q_d\}|=4}} 2\rho_{p_1 k}^2 = O(m^2 \|\mathbf{R} - \mathbf{I}_m\|_F^2);$$

for d' equals 1 or 2, it must be that

$$\begin{aligned}
 (D.15) \quad & \sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2 \\ |\cup_{d=1}^2 \{p_d, q_d\}|=4}} \mathbb{E} [(X_{p_1}^2 - 1)(X_{p_{d'}} X_{q_{d'}} - \rho_{p_{d'} q_{d'}})] \\
 &= \sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2 \\ |\cup_{d=1}^2 \{p_d, q_d\}|=4}} \rho_{p_1 q_{d'}} \rho_{p_1 p_{d'}} + \rho_{p_1 p_{d'}} \rho_{p_1 q_{d'}} = O(m^3) \|\mathbf{R} - \mathbf{I}_m\|_F.
 \end{aligned}$$

Substituting (D.14) and (D.15) into (D.13) gives that

$$\sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2 \\ |\cup_{d=1}^2 \{p_d, q_d\}|=4}} \mathbb{E} \left[\prod_{d=1}^2 \bar{S}_{p_d p_d} \bar{S}_{q_d q_d} \bar{S}_{p_d q_d}^2 \right] = O \left(\frac{m^4}{n^5} + \frac{m^2 \|\mathbf{R} - \mathbf{I}_m\|_F^2}{n^4} + \frac{m^3 \|\mathbf{R} - \mathbf{I}_m\|_F}{n^4} \right)$$

which gives us an estimate less than the one required. \square

Proof of (4.13) and Lemma 4.3. As described by the main text, with the help of the **Expectation** function provided by **mathematica**, we easily find that

$$\begin{aligned}
 \mathbb{E}[h_{1,pq}] &= \frac{4 \binom{n}{4}}{n^2 \binom{n-1}{3}} (1 + \rho_{pq}^2) = \frac{1 + \rho_{pq}^2}{n}, \\
 \mathbb{E}[h_{2,pq}] &= \mathbb{E}[\bar{h}_{2,pq}] = \frac{8 \binom{n}{4}}{n^3 \binom{n-1}{3}} (1 + 3\rho_{pq}^2) = \frac{2(1 + 3\rho_{pq}^2)}{n^2} \quad \text{and} \\
 \mathbb{E}[h_{3,pq}] &= \mathbb{E}[\bar{h}_{3,pq}] = \left(\frac{24}{n^4 \binom{n-2}{2}} + \frac{40}{\binom{n-1}{3} n^4} \right) \binom{n}{4} (1 + 5\rho_{pq}^2) = \frac{2(4+n)(1 + 5\rho_{pq}^2)}{n^3}
 \end{aligned}$$

for each pair $1 \leq p < q \leq m$. Collecting these and summing over all $1 \leq p < q \leq m$ gives the expectation of the kernel h in (4.13). We will now prove Lemma 4.3, first dealing with (4.19). Note that $g_4(\cdot)$ simply equals the kernel function h , in

particular for a set of distinct sample indices $i, j, k, l \in [n]$ we have

$$\begin{aligned}
g_4(\mathbf{X}_i, \mathbf{X}_j, \mathbf{X}_k, \mathbf{X}_l) &= h(\mathbf{X}_i, \mathbf{X}_j, \mathbf{X}_k, \mathbf{X}_l) \\
&= O(1) \times \\
&\quad \sum_{1 \leq p < q \leq m} \\
&\quad \left\{ - \sum_{\substack{\{i', j', k'\} \\ \subset \{i, j, k, l\} \\ i', j', k' \text{ distinct} \\ \text{and unordered}}} \sum_{\pi \in \mathcal{S}_3} \left[(X_{p\pi(i')}^2 - 1)(X_{p\pi(j')}X_{q\pi(j')} - \rho_{pq})(X_{p\pi(k')}X_{q\pi(k')} - \rho_{pq}) \right] \right. \\
&\quad - \sum_{\substack{\{i', j', k'\} \\ \subset \{i, j, k, l\} \\ i', j', k' \text{ distinct} \\ \text{and unordered}}} \sum_{\pi \in \mathcal{S}_3} \left[(X_{q\pi(i')}^2 - 1)(X_{p\pi(j')}X_{q\pi(j')} - \rho_{pq})(X_{p\pi(k')}X_{q\pi(k')} - \rho_{pq}) \right] \\
&\quad + \sum_{\pi \in \mathcal{S}_4} \left[(X_{p\pi(i)}^2 - 1)(X_{p\pi(j)}^2 - 1)(X_{p\pi(k)}X_{q\pi(k)} - \rho_{pq})(X_{p\pi(l)}X_{q\pi(l)} - \rho_{pq}) \right] \\
&\quad + \sum_{\pi \in \mathcal{S}_4} \left[(X_{q\pi(i)}^2 - 1)(X_{q\pi(j)}^2 - 1)(X_{p\pi(k)}X_{q\pi(k)} - \rho_{pq})(X_{p\pi(l)}X_{q\pi(l)} - \rho_{pq}) \right] \Big\} \\
&\quad + \sum_{1 \leq p < q \leq m} t_4(\mathbf{X}_{pq,i}, \mathbf{X}_{pq,j}, \mathbf{X}_{pq,k}, \mathbf{X}_{pq,l}, \rho_{pq}) O(n^{-1}) \\
&:= \tilde{g}_4(\mathbf{X}_i, \mathbf{X}_j, \mathbf{X}_k, \mathbf{X}_l) + \sum_{1 \leq p < q \leq m} t_4(\mathbf{X}_{pq,i}, \mathbf{X}_{pq,j}, \mathbf{X}_{pq,k}, \mathbf{X}_{pq,l}, \rho_{pq}) O(n^{-1}),
\end{aligned}$$

by collecting the $O(1)$ terms in the definition of $h_{2,pq}, \bar{h}_{2,pq}, h_{3,pq}, \bar{h}_{3,pq}$, where $t_4(\cdot)$ is just a fixed polynomial in $\mathbf{X}_{pq,i}, \mathbf{X}_{pq,j}, \mathbf{X}_{pq,k}, \mathbf{X}_{pq,l}, \rho_{pq}$ whose form is irrelevant to us. Using the fact that $2|ab| \leq a^2 + b^2$ for all $a, b \in \mathbb{R}$, we have

$$(D.16) \quad \zeta_4 \lesssim \mathbb{E}[\tilde{g}_4(\mathbf{X}_i, \mathbf{X}_j, \mathbf{X}_k, \mathbf{X}_l)^2] + O\left(\frac{m^4}{n^2}\right).$$

A key observation is that upon squaring and taking expectation, $\mathbb{E}[\tilde{g}_4(\mathbf{X}_i, \mathbf{X}_j, \mathbf{X}_k, \mathbf{X}_l)^2]$ must be a sum of finitely many terms, where for some sample indices $\tilde{i}, \tilde{j} \in \{i, j, k, l\}$, each of these terms can be “ \lesssim ” bounded by the form

$$(D.17) \quad \sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2}} \mathbb{E}[A(p_1 q_1, \tilde{i}) B(p_2 q_2, \tilde{j})],$$

where for any sample index $i \in [n]$ and variable indice $p, q \in [m]$, $A(p, q, i)$ and $B(p, q, i)$ may equal to one of

$$X_{pi}^2 - 1, \quad X_{qi}^2 - 1, \quad X_{pi}X_{qi} - \rho_{pq}.$$

Now if $\tilde{i} \neq \tilde{j}$, the form in (D.17) equals zero. If $\tilde{i} = \tilde{j}$, the form in (D.17) can be bounded by

$$\sum_{\substack{1 \leq p_d < q_d \leq m \\ d=1,2 \\ |\cup_{d=1}^2 \{p_d, q_d\}|=4}} \mathbb{E}[A(p_1 q_1, \tilde{i}) B(p_2 q_2, \tilde{i})] + O(m^3),$$

and by applying Corollary B.3, we leave it for the reader to check that the leading term in the preceding display must be “ \lesssim ” bounded by $m^3 \|\mathbf{R} - \mathbf{I}_m\|_F$. Summarizing this gives us the bound in (4.19).

Now we get to (4.16)-(4.18). The functions $g_c(\cdot)$, $c = 1, \dots, 3$ for the kernel h can be found by simply conditioning and taking expectation using Theorem B.2. With the help of `mathematica`, they are found to be

$$\begin{aligned} & g_1(\mathbf{X}_i) \\ &= \sum_{p < q} \frac{1}{4n} \{ (X_{pi} X_{qi} - \rho_{pq})^2 - (1 + \rho_{pq}^2) \} - \\ & \quad \sum_{p < q} \frac{1}{4n} \{ (1 + \rho_{pq}^2)(X_{pi}^2 + X_{qi}^2 - 2) + 8\rho_{pq}(X_{pi} X_{qi} - \rho_{pq}) \} + \\ & \quad \sum_{p < q} t_1(X_{pi}, X_{qi}, \rho_{pq}) O(n^{-2}) \\ &:= \tilde{g}_1(\mathbf{X}_i) + \sum_{p < q} t_1(X_{pi}, X_{qi}, \rho_{pq}) O(n^{-2}), \end{aligned}$$

$$\begin{aligned}
& g_2(\mathbf{X}_i, \mathbf{X}_j) \\
&= \frac{1}{4n} \sum_{p < q} \{ (X_{pi}X_{qi} - \rho_{pq})^2 + (X_{pj}X_{qj} - \rho_{pq})^2 - 2(1 + \rho_{pq}^2) \} - \\
& \quad \frac{1}{12n} \sum_{p < q} \left\{ \left[\sum_{\pi \in \mathcal{S}_2} (X_{p\pi(i)}^2 - 1)(X_{p\pi(j)}X_{q\pi(j)} - \rho_{pq})^2 + \right. \right. \\
& \quad \left. \left. 2(X_{p\pi(i)}^2 - 1)(X_{p\pi(i)}X_{q\pi(i)} - \rho_{pq})(X_{p\pi(j)}X_{q\pi(j)} - \rho_{pq}) \right] + \right. \\
& \quad \left. 2[(1 + \rho_{pq}^2)(X_{pi}^2 + X_{pj}^2 - 2) + 4\rho_{pq}(X_{pi}X_{qi} + X_{pj}X_{qj} - 2\rho_{pq})] \right\} - \\
& \quad \frac{1}{12n} \sum_{p < q} \left\{ \left[\sum_{\pi \in \mathcal{S}_2} (X_{q\pi(i)}^2 - 1)(X_{p\pi(j)}X_{q\pi(j)} - \rho_{pq})^2 + \right. \right. \\
& \quad \left. \left. 2(X_{q\pi(i)}^2 - 1)(X_{p\pi(i)}X_{q\pi(i)} - \rho_{pq})(X_{p\pi(j)}X_{q\pi(j)} - \rho_{pq}) \right] + \right. \\
& \quad \left. 2[(1 + \rho_{pq}^2)(X_{qi}^2 + X_{qj}^2 - 2) + 4\rho_{pq}(X_{pi}X_{qi} + X_{pj}X_{qj} - 2\rho_{pq})] \right\} + \\
& \quad \frac{1}{12n} \sum_{p < q} \left\{ 4(X_{pi}X_{qi} - \rho_{pq})(X_{pj}X_{qj} - \rho_{pq}) \right. \\
& \quad \left. + 2(1 + \rho_{pq}^2)(X_{pi}^2 - 1)(X_{pj}^2 - 1) \right. \\
& \quad \left. + 8\rho_{pq}[(X_{pi}^2 - 1)(X_{pj}X_{qj} - \rho_{pq}) + (X_{pj}^2 - 1)(X_{pi}X_{qi} - \rho_{pq})] \right\} + \\
& \quad \frac{1}{12n} \sum_{p < q} \left\{ 4(X_{pi}X_{qi} - \rho_{pq})(X_{pj}X_{qj} - \rho_{pq}) \right. \\
& \quad \left. + 2(1 + \rho_{pq}^2)(X_{qi}^2 - 1)(X_{qj}^2 - 1) \right. \\
& \quad \left. + 8\rho_{pq}[(X_{qi}^2 - 1)(X_{pj}X_{qj} - \rho_{pq}) + (X_{qj}^2 - 1)(X_{pi}X_{qi} - \rho_{pq})] \right\} + \\
& \quad O(n^{-2}) \sum_{p < q} t_2(X_{pi}, X_{qi}, X_{pj}, X_{qj}, \rho_{pq}) \\
&:= \tilde{g}_2(\mathbf{X}_i, \mathbf{X}_j) + O(n^{-2}) \sum_{p < q} t_2(X_{pi}, X_{qi}, X_{pj}, X_{qj}, \rho_{pq})
\end{aligned}$$

and

$$\begin{aligned}
& g_3(\mathbf{X}_i, \mathbf{X}_j, \mathbf{X}_k) \\
&= \sum_{p < q} \frac{-1}{12} \sum_{\pi \in \mathcal{S}_3} (X_{p\pi(i)}^2 - 1)(X_{p\pi(j)}X_{q\pi(j)} - \rho_{pq})(X_{p\pi(k)}X_{q\pi(k)} - \rho_{pq}) + \\
& \quad \sum_{p < q} \frac{-1}{12} \sum_{\pi \in \mathcal{S}_3} (X_{q\pi(i)}^2 - 1)(X_{p\pi(j)}X_{q\pi(j)} - \rho_{pq})(X_{p\pi(k)}X_{q\pi(k)} - \rho_{pq}) + \\
& \quad O(n^{-1}) \sum_{1 \leq p < q \leq m} t_3(X_{pi}, X_{qi}, X_{pj}, X_{qj}, X_{pk}, X_{qk}, \rho_{pq}) \\
&:= \tilde{g}_3(\mathbf{X}_i, \mathbf{X}_j, \mathbf{X}_k) + O(n^{-1}) \sum_{1 \leq p < q \leq m} t_3(X_{pi}, X_{qi}, X_{pj}, X_{qj}, X_{pk}, X_{qk}, \rho_{pq})
\end{aligned}$$

Above, $t_1(\cdot)$, $t_2(\cdot)$ and $t_3(\cdot)$ are simply three fixed polynomials in their respective arguments, and their forms are irrelevant to us. $\tilde{g}_1(\cdot)$, $\tilde{g}_2(\cdot)$ and $\tilde{g}_3(\cdot)$ simply collect

the terms that do not involve $t_1(\cdot)$, $t_2(\cdot)$ and $t_3(\cdot)$, respectively. Note that by the same fact that $2|ab| \leq a^2 + b^2$ for $a, b \in \mathbb{R}$,

$$(D.18) \quad \zeta_1 \lesssim \mathbb{E} [\tilde{g}_1(\mathbf{X}_i)^2] + O\left(\frac{m^4}{n^4}\right),$$

$$(D.19) \quad \zeta_2 \lesssim \mathbb{E} [\tilde{g}_2(\mathbf{X}_i, \mathbf{X}_j)^2] + O\left(\frac{m^4}{n^4}\right),$$

$$(D.20) \quad \zeta_3 \lesssim \mathbb{E} [\tilde{g}_3(\mathbf{X}_i, \mathbf{X}_j, \mathbf{X}_k)^2] + O\left(\frac{m^4}{n^2}\right).$$

Note that in the definition of \tilde{g}_1 , there is a leading factor of order n^{-1} . By applying Theorem B.2 with the help of `mathematica`, we find that $\mathbb{E}[\tilde{g}_1(\mathbf{X}_i)^2]$ is a finite sum of terms each, up to a factor of order n^{-2} , can be bounded by one of the forms:

$$(D.21) \quad \sum_{1 \leq r < s \leq m} \sum_{1 \leq p < q \leq m} \rho_{pq}^2, \quad \sum_{1 \leq p, q, r, s \leq m} \rho_{pq}^2 \rho_{rs}^2, \quad \sum_{1 \leq p, q, r, s \leq m} |\rho_{pq} \rho_{rs} \rho_{pr} \rho_{qs}|.$$

We leave it for the reader to check that with the two estimates in Lemma B.5 and the familiar trick of decomposing a sum according to the cardinality of an index set as in (3.5), the forms in (D.21) can all be bounded by

$$\|\mathbf{R} - \mathbf{I}_m\|_F^4 + m^2(1 + \|\mathbf{R} - \mathbf{I}_m\|_F^2)$$

up to constants, and hence from (D.18) we obtain the estimate for ζ_1 in (4.16). By the same token, with the help of `Mathematica` we observe that

$$\begin{aligned} \mathbb{E} [\tilde{g}_2(\mathbf{X}_i, \mathbf{X}_j)^2] &\lesssim O(n^{-2}) \sum_{1 \leq p, q, r, s \leq m} |\rho_{pq}| \\ \mathbb{E} [(\tilde{g}_3(\mathbf{X}_i, \mathbf{X}_j, \mathbf{X}_k))^2] &\lesssim \sum_{1 \leq p, q, r, s \leq m} \rho_{pq}^2 \rho_{rs}^2 + \sum_{1 \leq p, q, r, s \leq m} |\rho_{pq} \rho_{rs} \rho_{pr} \rho_{qs}|, \end{aligned}$$

again by the index set decomposition trick and Lemma B.5 we have

$$(D.22) \quad \mathbb{E} [\tilde{g}_2(\mathbf{X}_i, \mathbf{X}_j)^2] \lesssim n^{-2} m^3 (\|\mathbf{R} - \mathbf{I}_m\|_F + 1) \text{ and}$$

$$(D.23) \quad \mathbb{E} [(\tilde{g}_3(\mathbf{X}_i, \mathbf{X}_j, \mathbf{X}_k))^2] \lesssim \|\mathbf{R} - \mathbf{I}_m\|_F^4 + m^2(1 + \|\mathbf{R} - \mathbf{I}_m\|_F^2).$$

Collecting (D.19), (D.20), (D.22) and (D.23) gives us (4.17) and (4.18). \square

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